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International Trends in Post-Primary Mathematics Education:

Perspectives on Learning, Teaching and Assessment

Paul F. Conway and Finbarr C. Sloane

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Paul F. Conway and Finbarr C. Sloane

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1.1 Introduction

An ideal education in which students have democratic access to powerful mathematical ideas can result in students having the mathematical skills, knowledge and understanding to become educated citizens who use their political rights to shape government and their personal futures. They see the power of mathematics and understand that they can use mathematical power to address ills in our society.

C. E. Malloy, Democratic Access to Mathematics through Democratic Education, 2002

Many important new developments have a mathematical basis. Documents and data sent electronically need to be securely and accurately transmitted. Cryptography is the science of the secret transmission of data, used nowadays on the Internet, by financial institutions and many others. Coding theory is used for the accurate transmission of data - error correcting codes are used, for example, in CDs. Formal methods provide a rigorous mathematical basis to software development and must be used for safety critical systems - needed in the aviation, military, medical and other fields. Storage, compression and recovery of large amounts of data are achieved using mathematical transforms - wavelet transforms are used to store fingerprints on FBI computers. Statistics and probability are an important part in the study of networking. The graphical images in Microsoft's Encarta encyclopaedia are stored in highly compressed form on the CD by means of techniques developed by mathematicians working in fractal geometry. Newer developments in the financial services sector need highly numerate and computer literate graduates. Neural networks are mathematical/computer models with a learning process that try to mimic the workings of the human brain. These have found extensive applications in many areas, as in the prediction and modelling of markets, signature analysis, selection of investments. Quote from a recent article: "Neural networking is the buzz word in the insurance industry."

Ted Hurley, Prof. of Mathematics, UCG, 'Computers need Maths', Irish Times, 4 February 1997

The study of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest…Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it…The reason for this failure of the science to live up to its great reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception (p. 1-2).

Alfred North Whitehead, An Introduction to Mathematics (1911)

…it has been widely recognised that mathematics occupies a rather special position. It is a major intellectual discipline in its own right, as well as providing the underpinning language for the rest of science and engineering and, increasingly, for other disciplines in the social and medical sciences. It underpins major sectors of modern business and industry, in particular, financial services and ICT. It also provides the individual citizen with empowering skills for the conduct of private and social life and with key skills required at virtually all levels of employment.

Making Mathematics Count: The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education, Roberts Report, United Kingdom (2004)

As articulated in Hurley's *Irish Times* article, an understanding of how mathematics is so crucial, and often central, to contemporary scientific developments underpins much of the ongoing drive to enhance mathematics education in the current wave of globalisation. This point of view could be characterised as 'mathematics for scientific advancement'. This 'mathematics for scientific advancement' position, together with the idea of 'mathematics for all', a democratic citizenship argument (see Malloy above), together represent two key forces that today drive the formulation of mathematics education policies. Consequently, few meetings about educational reform in any part of the world conclude without an endorsement of the central role that mathematics can and ought to play in education; the fruits of high-quality mathematics education, it is asserted, ensure a rich harvest for the economy and society. As such, mathematics is perceived as a high-yield school subject, especially so now in the current wave of globalisation with its attendant pressures on many economies to move up the value chain through the development of a 'high skill society' (Brown, Green, and Lauder, 2001). Despite what seems to be the unassailable position of those advocating enhanced mathematics education, either from the 'mathematics for all' or the scientific advancement positions mathematics is often perceived, as Whitehead so clearly articulates, as an impenetrable and abstract subject, and furthermore the teaching of it sometimes fosters such views. Thus, mathematics is seen simultaneously as increasingly important yet also as an especially difficult subject, despite its wondrous applications and essential role in today's world. Indeed, understanding our world today without the conceptual tools provided by mathematics would be unimaginable.

Mathematics, or more properly mathematical sciences, have played, it is argued, an important role as a powerful symbolic system leading and contributing to world-changing innovations in society, such as the development of new technologies, and a means of conveying in a powerful and compelling form significant applied and theoretical insights across a range of disciplines and professional fields.

1.2 Overview of research report

In the context of the review by the National Council for Curriculum and Assessment (NCCA) of mathematics education at post-primary level in Ireland, this study focuses on a number of key international trends in mathematics education. While it is cognisant of developments in Ireland, it does not focus on Irish developments per se. However, in the final chapter we note potential areas of overlap, areas where there are significant divergences and possible lines of development in mathematics education at post-primary level in Ireland.

The NCCA's companion document provides a review of developments in the Irish context (NCCA, April, 2005). In reviewing international trends in terms of their implications for both curriculum and assessment, we highlight selected countries and specific initiatives, and concentrate on a few key ideas: *new insights on mathematics teaching* from recent video studies; *new perspectives on learning and their implications for designing powerful learning environments;* and *conditions for high quality teacher education in mathematics* (including initial, early career and continuing professional development). Thus, we have chosen to focus on a few big ideas rather than adopting the encyclopaedic approach of numerous handbooks on mathematics published over the last decade (Grouws, 1992; English, 2002). In summary, the purpose of this report is to highlight the most significant trends in mathematics education internationally that might inform the current review of mathematics at post-primary level in Ireland.

Drawing on research in the learning sciences (Collins, Greeno and Resnick, 1996; Bransford, Brown, and Cocking, 2000), the study looks at developments in mathematics education within dominant approaches to learning as they pertain to current debates on the development of 'powerful learning environments' (De Corte *et al.*, 2003). The study is organised into five chapters. Chapter 1 will frame the study in terms of key concerns, images, and provision in relation to mathematics internationally. Chapter 1 provides an outline of the proposed study and reflects the outcome of consultation between the researchers and the NCCA with regard to the focus and content of this report, discusses the significance of mathematics education as an increasingly important curricular priority internationally, and describes key developments in relation to provision, curriculum and assessment in selected countries. As the current review of mathematics education in Ireland is focused on post-primary education, we look at the specific and changing educational goals and societal context of this phase of education. One of the main points we note in this first chapter is the manner in which countries are engaging in various cross-national efforts to enhance mathematics education.

Chapters 2 and 3 will address frameworks and philosophies that underpin mathematics education internationally, such as the realistic mathematics education movement and the concept of 'mathematical literacy' for all. Chapters 2 and 3 provide a review and analysis of two key trends in mathematics education internationally: (i) the cross-national efforts to understand the dynamics of mathematics teaching, and (ii) developments in the learning sciences, including their actual and potential impact on mathematics education. For the purposes of this report, Chapter 2 focuses on developments in understanding the teaching of mathematics, drawing primarily on compelling findings from the TIMSS 1995 and TIMSS-R 1999 video studies. As one of the chief insights of both video studies was the nature of culture-specific 'lesson signatures', the second section of Chapter 2 provides a review of developments in Japan and elsewhere in relation to lesson study as a form of professional development for teachers and what it suggests about curricular reform.

Chapter 3 addresses the impact of insights from research into the learning sciences on mathematics education, focusing on the current interest in brain-based research, the 'social' turn in learning theories, and the increasing prominence of self-regulated learning as a policy and research priority. The first section of the chapter provides an overview of learning sciences research and its role in understanding, designing and evaluating powerful learning environments in mathematics education. In examining behavioural, cognitive and socio-cultural/situated perspectives on learning, we note their impact, current status and potential to inform mathematics education. This section also describes the significant contribution of Han Freudenthal's views on the teaching and learning of mathematics and their influence on and relevance to contemporary debates on mathematics education. The subsequent section of this chapter outlines key developments in cognitive neuroscience and debates about the utility of such knowledge for cognitive psychology as a conceptual lens on the teaching and learning of mathematics.

Cognitive neuroscience has been the subject of considerable debate in education over the last decade; the OECD and US National Research Council among others have been examining its potential implications for education, especially literacy and numeracy. The final section of Chapter 3 reviews recent work on self-regulated learning (SRL), and considers its appeal to policy makers, researchers and reformers involved with mathematics education internationally, in a context where lifelong learning is becoming a priority. Building on Chapters 2 and 3, Chapter 4 will address key initiatives in mathematics education internationally to the extent that they illuminate issues of relevance to Irish post-primary mathematics education.

Chapter 5, the final chapter, notes key issues arising from previous chapters with a focus on issues that might form the basis for some of the discussion in the current review of post-primary mathematics education. This concluding section is framed within current curriculum developments in Irish post-primary education and discussions in Ireland of the 2003 PISA study of mathematics literacy (see Cosgrove *et al*., 2005), a video study of junior cycle mathematics education (Lyons *et al*., 2003) and recent research presented at the first Mathematics Education in Ireland conference (Close, Corcoran and Dooley, 2005).

1.3 What is mathematics?

> *A widespread public image of mathematics is that it is a difficult, cold, abstract, theoretical, ultra-rational but important and largely masculine. It also has an image of being remote and inaccessible to all but a few super-intelligent beings with 'mathematical minds'.*
>
> **(Ernest, 2004, p. 11)**

> *To most people, mathematics means working with numbers….this definition has been out of date for nearly 2,500 years. Mathematicians now see their work as the study of patterns real or imagined, visual or mental, arising from the natural world or from within the human mind.*
>
> **(Devlin, 1997, Mathematics: The Science of Patterns)**

In a classic text on the history of mathematics, Struik (1948) recognises and acknowledges reciprocal relationships between developments in mathematics and its historical and cultural contexts. In coming to an understanding of its origins, scope and influences, Struik reminds us that 'mathematics has been influenced by agriculture, commerce, and manufacture, by warfare, engineering, and philosophy, by physics, and astronomy' (p. 1). Furthermore, he notes that 'The influence of hydrodynamics on function theory, of Kantianism and surveying on geometry, of electromagnetism on differential equations; of Cartesianism on mechanics, and of scholasticism on calculus' (p. 1) all point to both the culturally-influenced nature of mathematics and the scope of mathematical ways of knowing. Struik's history of mathematics ends in 1945 because, he claims, 'the mathematics of the last decades of the twentieth century has so many aspects that it is impossible… to do justice even to the main trends' (p. 1). Tracing the origins of contemporary mathematics, over the last fifteen millennia, through the contributions of and affinity between Egyptian, Babylonian, Chinese, Indian, Arabian, Greek and other mathematical traditions (e.g. the less influential mathematical traditions of the Minoan-Mycenaeans, the Mayas and the Incas), Struik portrays mathematics as a dynamic and increasingly diverse field of knowledge influenced by but also influencing society in far-reaching ways. For example, he notes that 15,000 year-old cave paintings in France and Spain 'reveal a remarkable understanding of form; mathematically speaking, they reveal understanding of two-dimensional mapping of objects in space' (p. 9). These cave paintings were undertaken, as well as the more recent physical constructions in places such as Stonehenge in England, the pyramids in Egypt, or Newgrange in Ireland, point to sophisticated and relatively independent astronomical and mathematical knowledge traditions existing in different parts of the world. This brief reference to the history of mathematics illustrates one key position we adopt in this report - namely, the changing, dynamic and diverse nature of mathematical ways of knowing. As such, it reminds us that rather than being viewed as a timeless edifice, mathematics might more accurately be viewed as a complex growing tree. Indeed, the image of a tree has appealed to some mathematics educators (Romberg and Kaput, 1999). For example, Thurston's banyan tree metaphor has gained some adherents:

> *Mathematics isn't a palm tree, with a single long straight trunk covered with scratchy formulas. It's a banyan tree with many interconnected trunks and branches - a banyan tree that has grown to the size of a forest, inviting us to climb and explore. (p. 7)*

However, even adopting a historical perspective on the development of mathematics may not convey the current dynamic nature of evolving mathematical knowledge, in that it may be tempting to see historical aspects of mathematics as interesting and curious *ways of knowing* before the advent of a robust and relatively unchanging domain of mathematical knowledge in the more recent past.

The banyan tree image seems, at one level, overly abstract and, at another level, vivid and compelling. It clearly conveys a broad-ranging and expansive view of mathematics but in conveying a such a view, it could be argued that it might draw education and educators away from a focus on 'core' mathematical competences needed for living and learning in the 21st century. Indeed, over the last fifteen years there have been debates raging both nationally and internationally about the state and scope of mathematics education (and other curricular areas) in view of the perceived 'high skill' challenges of the 21st century. Debates about the state of mathematics education tend to focus on students' overall poor understanding of key mathematical concepts and procedures, typically reflected in routine rather than flexible use of mathematical ideas (APEC, 2004; Bramall and White, 2000; De Corte *et al.*, 1996;

Gardner, 1991). This has led to a world-wide drive to improve students' mathematical understanding. While the goal of teaching for mathematical understanding has been a goal for at least a hundred years (Romberg and Kaput, 1999), it has taken on a new meaning and urgency in current debates on the quality of mathematics education. These debates on the quality of mathematics education are occurring within the context of a wider global debate on the quality of educational outcomes (UNESCO, 2005; OECD, 2005). For example, in the European Union, the *Work Programme for 2010* has created a new tempo for European educational policies to focus on educational quality in order to meet the goals of Lisbon 2000 (Novoa and DeJong-Lambert, 2003; Conway, 2005a).

Debates about the scope of mathematics have been focusing on the extent to which school mathematics curricula are disconnected from real-world contexts and applications (De Corte *et al,* 1996), and the degree to which mathematics curricula ought to emphasise numeracy vis-á-vis the broader expanse of the mathematics curriculum; New Zealand's Numeracy Project[1] at post-primary level, for example, has been grappling with this issue. The current discussion, occurring in the context of the somewhat different foci of the Third International Mathematics and Science Study (TIMSS) and the OECD's PISA international comparative studies, captures this concern about the extent to which school mathematics is sufficiently connected to real-world contexts. In the IEA tradition, TIMSS and TIMSS-R tests were based on careful mapping of curricula across participating countries, whereas the PISA mathematical literacy studies are based on an agreed understanding of the type of *in and out-of-school mathematics* that students need to know for life and work in the 21st century. This has resulted in different focuses in the

[1] "The focus of the Numeracy Development Project is improving student performance in mathematics through improving the professional capability of teachers. It is intended that by 2005, almost every New Zealand teacher of year 1 to 3 children and many teachers of year 4 to 8 children will have had the opportunity to participate". http://www.nzmaths.co.nz/Numeracy/Intro.aspx.

TIMSS and PISA mathematics tests. The issues raised about both the content and format of mathematics tasks, surfacing in the different emphases of major and influential comparative studies, reflect a critical debate at the heart of mathematics education about the nature and scope of mathematics itself, and also of what counts as worthwhile mathematics for inclusion in school curricula and syllabi world-wide. For example, during 2005 in China, there have been 'math wars' a decade after the USA National Council of Teachers of Mathematics (NCTM)-modelled reform of mathematics curricula. As Zhao notes:

> *At the annual session of the Chinese National People's Congress (NPC), a group of members of the Chinese legislative body introduced a proposal calling for an immediate overhaul of the New Mathematics Curriculum Standards for elementary and secondary schools… The New Math Curriculum has been criticised for betraying the excellent education tradition, sacrificing mathematical thinking and reasoning for experiential learning, giving up disciplinary coherence in the name of inquiry learning, lowering expectations in the name of reducing student burden, and causing confusion among teachers and students. (2005, p. 1).*

According to Zhao, similar views have been voiced by leading Ministry of Education officials, parents, mathematicians, scientists and textbook publishers (see also S. J. Wang, 2005).

The debate in China over the last year is a good example of the internationalisation of mathematics education that has occurred in the field over the last forty years. As de Corte *et al.* note:

> *A further important development is the importance that the international dimension in mathematics education has come to assume in the last 20 years. Among the manifestations of this trend may be listed the growth of*

> *contact, collaboration, and international institutions; the carrying out of international comparisons of children's mathematical performance; and a more balanced recognition of the mathematical achievements of all cultures. (1996, p. 456)*

The International Congress for Mathematics Education (ICME) has provided a forum for mathematics educators and other educational researchers working in mathematics to share ideas, and has ensured a flow of information and key ideas in mathematics education across national borders and across otherwise relatively impermeable political boundaries. For example, Hans Freudenthal was invited to China in the late 1980s to give a series of lectures on Realistic Mathematics Education (RME) in order to inform mathematics education reform proposals.

Mathematics as a 'basic' skill: three views

The focus on the 'real world' credentials of school mathematics overlaps in part with a debate about mathematics as a 'basic' skill in the sense that if maths is real it provides a basic, well-grounded or fundamental preparation for the future. Policy-wise and logically a strong and plausible case can be made for viewing mathematics as a basic skill. The policy and logic argument might run as follows: mathematics is a powerful symbolic system in itself and forms an essential foundation for a wide range of other disciplines and professional fields essential in fostering a 'high skill' society (see, for example, the opening quotations in this report). As such, it is 'basic' or foundational in that in its absence the economy and society would suffer in not having mathematically skilled workers and citizens ready to deploy and advance on these basic skills. This, however, is only one of the ways in which the term 'basic' is used to portray mathematics as important in a foundational sense (see the two collections of edited essays *Why learn maths?* [2000, Bramall and White] and *The maths we need now* [Tikly and Wolf, 2000] which discuss the role of mathematics and mathematics education in contemporary society).

A second and more problematic use of the term 'basic' in mathematics education comes from an epistemological and/or learning perspective. The argument runs as follows: mathematics - and especially core operations, concepts and procedures - can be ordered sequentially, and lines of development for learners can be specified reflecting a move from simple to more complex mathematics over time, as well as over the cycle of students' school careers. On the face of it, this argument seems compelling. Clearly, it is argued, some mathematics tasks are easier than others (complexity); clearly, it might be useful pedagogically to order operations, concepts and procedures from simple to complex (pedagogy); and clearly, mathematics, more than any other subject, reflects agreed-upon or high-consensus content (epistemology). We do not address all of these issues here, save to note that research into learning has demonstrated how even so-called 'basic' skills such as reading, writing and mathematics, even at their most elementary levels, involve complex and multi-dimensional cognitive and cultural factors (Means, Chelemer and Knapp, 1991). Means and Knapp note that cognitive science has provided ample evidence to question the logic that students should be given certain 'basic' skills in reading, writing and mathematics long before being exposed to more 'advanced' skills such as reading comprehension, writing composition and mathematical reasoning[2].

2 The argument we make here about reframing the relationship between what are termed basic and higher order skills in mathematics is very similar to the case made by some in science education that basic skills be taught in the context of investigations. (See for example Bennett, Lubben and Hogarth's [2003], review of context-based and Science-Technology-Society (STS) approaches in the teaching of secondary science.)

They note that:

> *By discarding assumptions about skill hierarchies and attempting to understand children's competencies as constructed and evolving within and outside of school, researchers are developing models of intervention that start with what children know and expose them to what has traditionally been thought of as higher-order thinking. (Means, Chelemer and Knapp, 1991, p. 181)*

Consequently, this second use of the term 'basic' to characterise mathematics can be problematic in optimally positioning mathematics within learning because it results in inappropriate skill hierarchies informing approaches to teaching; in curriculum because it misconstrues the relationships between different curriculum components; and in education policy debates because it can be used to argue for a *basics-first-and-advanced-skill-later* approach to guide the teaching of less able students (Means, Chelemer, and Knapp, 1991; Oakes and Lipton, 1998; Conway, 2002).

Finally, there is also a third sense in which 'basic' is often used, that is, to characterise the maths that is 'just enough' for weaker students, for students with special educational needs and for adults and returning learners. This use of the term basic emphasises the basic, functional and adaptive aspects of mathematics. As debates on mathematics unfold, it is important to distinguish between the various uses of 'basic' and the underlying assumptions of each.

1.4 Concerns about mathematics education

Internationally there are many concerns about the state of mathematics education. These concerns typically involve two sets of factors: that is, both *push* (poor levels of understanding and achievement gaps) and *pull* (the need for 21st century skills) factors.

In the case of *push* factors the focus tends to be on perceived poor levels of student understanding, as well as achievement gaps between boys and girls, between different ethnic groups, and between one country and another. Concerns about poor levels of understanding revolve around post-primary graduates' lack of capacity to apply mathematics in practical 'real world' contexts. Students' poor levels of mathematical understanding are typified by concerns about schools' focus on procedural, routine, inflexible, abstract, and inert knowledge rather than fostering students' capacity in conceptual, problem-focused, practical and flexible use of mathematical knowledge (de Corte, *et al.*, 1996; J. Wang, 2005). Summarising twenty years of research in the learning sciences, Gardner in his book *The Unschooled Mind* (1991) draws the attention of educational policy makers, educators and the wider public to the phenomenon whereby many degree holders do not have the capacity to apply their hard-earned knowledge appropriately in new contexts. He notes that there is now an extensive body of research examining this phenomenon in different disciplines and contexts and it clearly documents how those with advanced training are often unable to apply knowledge appropriately in new contexts or ill-structured problem domains. As Gardner notes,

> *Whether one looks at the physical sciences, the natural sciences, the human sciences, mathematics, history, or the arts, the same picture emerges: most students prove unable to master disciplinary content sufficiently so that they can apply it appropriately in new contexts. (p. 247)*

In an era of globalisation, with its increasingly rapid social, cultural and economic change, enhancing a system's as well as individuals' capacity to apply knowledge in new contexts is one of the key challenges education systems must address both within and across subject areas. In the context of mathematics, similar concerns have arisen based on observations of how students in school or after completion of school attempt to solve problems in routine fashion disregarding crucial information, especially contextual information (Verschaffel, Greer, and de Corte, 2000; Cooper and Harries, 2002).

A second long-standing push factor revolves around concerns about gender and ethnicity-related gaps in mathematical achievement. These concerns have been reflected in public debate, research and policy questions over many years. Historically, at least, concerns about gender differences in mathematical achievement have typically pointed out the consistent manner in which boys outperformed girls. More recently, the perceived underachievement of boys has become a source of concern across many countries, with girls now outperforming boys at the end of compulsory schooling in mathematics in many jurisdictions (see Elwood, 2005 for a detailed discussion of the role of examinations in the relationship between gender and achievement. Elwood's analysis focus on three aspects of examining: "styles of examinations and how they define achievement; coursework and the role it plays in contributing to gender differences in performance; and tiered entry systems in examinations and how they provide unequal opportunities for boys and girls to be successful", p.373).

Explanations about how to explain why boys outperformed girls historically revolve around two sets of factors: gender differences as deficiencies or hurdles (i.e. the spatial-visualization, mathematics anxiety, mathematics achievement, or the attribution patterns of girls) and social influences on mathematics participation (i.e. mathematics as a male domain, classroom culture, curriculum, and the manner in which teachers treated boys and girls differently). For example, the question as to whether there is a mathematics gene or not, and if boys were more likely to have it than girls, was the focus of a number of articles in the journal *Science* in the early 1980s (Benbow and Stanley, 1980; Kolata, 1981; Schafer and Gray, 1981). In line with the keen research and public interest in gender, genes and mathematics at that time, Kolata, for example, asked whether girls were born with less ability. The historical reality of girls' underachievement in mathematics has been and continues to be an issue in mathematics education in many countries. However, the nature of these concerns, and the related debates, are undergoing considerable change at present for reasons which include the increased opportunities for girls in mathematics and the more recent moral panic (mid-1990s onwards; see Skelton, 2001; Mac an Ghaill, Hanafin and Conway, 2004) about the perceived underperformance and low participation of some boys in mathematics - typically boys from poorer communities and/or low-status minority ethnic groups (Jacobs, Becker, Gilmer, 2001; Jencks and Phillips, 1998; Leder, 1992). Regardless of whether the concerns are about boys' or girls' perceived underachievement, they typically focus on the impact of stereotypical vocational choices and restricted labour market options on their future life-chances at personal, social and economic levels (Skelton, 2001).

In relation to ethnicity, the persistence of achievement gaps between ethnic groups has been a feature of debates in more ethnically diverse societies such as the USA, Canada, Australia and Germany. In the case of the USA, the perceived underperformance of Hispanic and African-American students in mathematics, compared to higher-performing Asian-American students, has led to extensive research on the cultural dimensions of mathematics achievement (Portes, 1996; Jencks and Phillips, 1998). Concerns about the underperformance of students from particular ethnic groups are typically part of wider debates on educational inequality, where achievement scores are one of the four primary indices of inequality, with years of education completed, grade retention and drop out or non-completion constituting the other three indices (Portes, 1996).

Ethnicity-related explanations of underachievement and inequality in mathematics and other subjects typically adopt on one of three positions: the cultural deficit, cultural difference, or post-cultural difference approaches (Portes, 1996). The cultural deficit approach 'draws attention to the adaptiveness in ethnic socialization practices and values relative to the norm, particularly in education' (Portes, 1996, p. 337). Cultural difference approaches focus on discontinuities between home and school cultures as reflected in different language practices, concepts and skills. Post-cultural difference models focus on one of two explanations: (i) double stratification of caste-like minorities and folk theory of success, or (ii) school-structured failure. The double stratification explanation focuses differences between the status of voluntary and involuntary minorities and subsequent school experiences of students from these distinctly different minorities (Ogbu, 1992). In general, voluntary minorities fare much better in the education system. For example, in the USA the very high performance of many students from East Asian immigrant voluntary minorities is contrasted with the underachievement in maths (and other subjects) of most African-American students who, as an involuntary minority, are subject to caste-like status rooted in a history of cultural and educational oppression. The second post-cultural difference explanation focuses on structural (e.g. curriculum, school organisation, textbooks) and/or functional (e.g. teaching practices and processes, assessments, time on task) that impede students from various ethnic groups in reaching their educational potential.

One of the distinctive features of recent analyses and commentaries on international comparative test results has been an intense media and research interest in understanding the reasons underlying the high performance of many East Asian countries. However, in their recent study of the role and conceptualisation of teacher knowledge in explaining the, more often than not, superior performance of China over the USA on international mathematics tests, Wang and Lin (2005) are very critical of the overly general categorisations of students from different East Asian cultural groups. Commenting on both the research studies and debates attempting to explain the cultural dimension of mathematics achievement, they note that:

> *…many are based on ambiguous cross-national categorization of East Asian students from Japan, China, Korea, and other East Asian regions and countries with little differentiation among them. Such ambiguous categorization is problematic considering that conceptual, institutional, and practical differences exist among those countries (Tobin, Wu, and Davidson, 1989). Similarly, U.S. students are often categorized as a homogeneous group without consideration of the similarities and differences among racial groups. For instance, findings on the performance of Asian Americans rarely distinguish among those whose ancestors are Japanese, Chinese, Korean, Vietnamese, Cambodian, Thai, Hmong, Laotian, Filipino, and so forth. Such broad categorizations may mask underlying ethnic and cultural differences and thus prevent adequate interpretation of differences related to student performance. (p. 4)*

Wang and Lin's comments and research point to the complexity involved in understanding the cultural dimension of mathematics achievement in an increasingly diverse world.

Cross-national studies in mathematics achievement tell a complicated story[3], raise many questions and tend to generate sporadic rather than

[3] For example, see O'Leary (2001) for a discussion of factors affecting the relative standing of countries in TIMSS.

sustained interest in educational matters. However, in many countries, the educational, public and media interest in the results of the IEA's TIMSS (1995), TIMSS-R (1999) and the OECD's Program for International Student Assessment (PISA) 2000 (where mathematics was a minor domain) and PISA 2003 (where mathematics was a major domain), have helped create and sustain a context in which national education systems are being held accountable to international standards in priority curricular areas. To cite a dramatic case, Germany's 'PISA-shock' in response to the mathematics literacy results in PISA 2003 is a good example of how international test results may have a wide ripple effect on public and educational opinion. In response to the perceived deficits in mathematics literacy in Germany, a new television programme on Saturday night, broadcast in a prime-time slot, presents PISA-like items in order to educate the public. In the USA, the *Christian Science Monitor*, in response to PISA 2003 results, had a headline 'Math + Test = Trouble for U.S. Economy'. The article noted that the performance of students on PISA's 'real-life' mathematics skill tests was a 'sobering' indicator of students' deficits on 'the kind of life-skills that employers care about.' Furthermore, comments by Ina Mullis, TIMSS co-director, about the USA's performance in TIMSS (2003), are indicative of a broad-based concern in the USA about its performance and ranking on international tests over the last decade: 'The United States has moved up a little, but there is still a huge gap between what our students can do and what students do in the highperforming Asian countries… Singapore has 44 percent of their students at the advanced level, while the United States has 7 percent' ('Math and Science Tests Find 4th and 8th Graders in U.S. Still Lag Many Peers', New York Times, 15 December 2004).

Moving beyond the performance of individual countries, international surveys in mathematics have complicated the interpretation of the gender dimension of test scores by illustrating how boys and girls differ in the manner in which they typically achieve their scores. Drawing on the results of IEA studies, Murphy (2001), for example, observed that, 'Girls' superior performance is associated with computation, whole number and estimation and approximation as well as algebra. Boys' superior performance is associated with geometry, particularly three-dimensional diagrams, measurement and proportional thinking' (p. 113). Situated within wider debates about the urgency of preparing students for knowledge societies in this era of globalisation, all of these analyses and commentaries point to the increasingly prominent international comparative context fuelling concerns about mathematics education.

With ever-increasing globalisation, educational policy-makers are identifying the new skills that students will need in order to prepare them to live in this century's knowledge society: the *pull* factor. Gardner (2004) classifies the knowledge, skills and dispositions needed in pre-collegiate education in an era of globalisation. They are:

- understanding the global system
- the ability to think analytically and creatively within disciplines
- the ability to tackle problem issues that do not respect disciplinary boundaries
- knowledge of other cultures and traditions as an end in itself and as a means to interact with people from other cultures
- knowledge of and respect for one's own cultural traditions
- fostering of hybrid or blended identities
- fostering of tolerance and appreciation across racial, linguistic, national and cultural boundaries.

In the context of mathematics education, some items on Gardner's list seem particularly relevant, such as the capacity for and importance of analytical and creative thinking, and problem solving within and across disciplinary boundaries.

1.5 Why is maths taught the way it is? Curricular cultures, textbooks and examinations/testing traditions

A wide variety of factors interact to shape the quality and content focus of the mathematics that is taught in schools. These include system-level features, wider community and social-cultural factors, approaches to teaching, and student characteristics such as motivation and ability. In this section, we focus on three system level features which profoundly shape the nature of students', teachers' and families' experiences of mathematics education: disciplinary/curricular culture in mathematics education, mathematics textbooks, and examination/testing traditions.

Curricular cultures in mathematics education: 'new/modern' maths and 'real world' maths

Two curricular cultures have been influential in mathematics education over the last forty years: a formal and abstraction-focused approach known as 'new' or 'modern' mathematics, and a more context-based, real-world and problem-focused mathematics education emanating from different sources including Piagetian constructivism, realistic mathematics education (RME) and situated cognition.

'New/modern' mathematics education

The 'new/modern' mathematics culture, especially influential in post-primary mathematics curricula of many countries since the early

1960s, has come under increasing scrutiny over the last decade or more. In Europe, it was given considerable support and impetus as an approach to mathematics education due to two main factors: an OECD-sponsored conference at Royaumont in 1959 (Goffree, 1993) and the ideas of the French Bourbaki school of mathematicians, whose abstract and logical views of mathematics had been percolating through the culture of university mathematics departments since the 1930s. In the USA, both a major conference at Woods Hole in 1959 and efforts to create a renewed focus on high-level mathematics in school curricula, in response to Russian's Sputnik achievements in space in 1958, culminated in a series of mathematics education reforms being identified, prominent among these the potential role of 'new math' (English and Halford, 1995). As a curricular culture, 'new' or 'modern' mathematics elevated abstraction as the most important value in mathematics education. This resulted in a formal, highly structured approach to teaching, with a focus on systematising mathematical ideas using mathematical symbols. As Goffree (1993) notes

> *Modern Mathematics is, proportionally speaking, a slight amount of knowledge with which you can do proportionally much more. Or, in other words, the more abstract mathematics is, the more applicable it can be. Abstract means: loose from context. (p. 29).*

English and Halford identify the key characteristics of the new math approach as a heavy emphasis on logical aspects of mathematics, explicit teaching of set theory, the laws of arithmetic, and Euclidean geometry. They note that these emphases were reflected in post-primary mathematics education textbooks' foci on set theory, systems of numeration and number theory. According to English and Halford (1995), 'The explicit teaching of the notation and algebra of sets and the general laws of arithmetic were by their very nature abstract' (pp. 7-8). As we noted above, much of the initial impetus for the development of new mathematics came from the universities through the impact of the Bourbaki school of mathematicians' who had been seeking to formulate a complete and integrated approach to mathematics. Their views had a profound impact on the philosophy of mathematics adopted by university mathematics departments in many countries from the 1930s right up to the present day, with far-reaching implications on the structure of the discipline as well as preferred approaches to teaching.

Mathematics in context: real-world problem-focused mathematics education

A second curricular culture, an approach to mathematics focused on real-world problem solving, has come to increasing prominence over the last two decades in mathematics education at second level in many countries. Its origins are less clear than the new mathematics movement and can be traced back to the influence of at least three major strands in mathematics education: Piagetian constructivism, realistic mathematics education (RME) and situated cognition. Two of these, Piagetian constructivism and situated cognition, have their roots in cognitive, educational and developmental psychology. RME grew out of the work of Hans Freudenthal, a Dutch-based mathematician turned mathematics educator, whose ideas have had a profound impact on mathematics education internationally. The importance of real-world contexts, as both a source of learning and site in which mathematical ideas can be applied, is perhaps the key idea uniting these different influences on mathematics education. Writing in the RME tradition, one of Freudenthal's colleagues notes that in RME '…reality does not only serve as an application area but also as the source for learning' (Goffree, 1993, p. 89). This position is in marked contrast to the new mathematics stance, whereby context is downgraded in favour of abstraction, and application is at best merely an add-on to abstraction.

Situated cognition and Piagetian constructivism share a number of features: a view of learners and learning as active; a view of knowledge as something constructed by learners in social and material contexts; and learners' sense-making as an important basis for the design of teaching and learning environments. Together with RME, they have now come to the forefront of thinking on mathematics education internationally, as evidenced by the OECD's adoption of situated cognition and realistic mathematics education as the bases for their preferred orientation to learning in the design of the PISA mathematics literacy study. In contrast with new/modern mathematics, the real world, context-focused mathematics education approach puts a premium on rich, socially-relevant contexts for learning, problem solving and the relevance of learners' prior experiences in planning and engaging with learners in classrooms or in multimedia learning environments (MMLEs). This move toward a more socially embedded view of mathematics reflects a wider change in the relationship between school and society as curricular knowledge is renegotiated in an era of globalisation.

"It's in the book": textbooks' role in shaping mathematics education

> *…a book is designed to give an authoritative pedagogic version of an area of knowledge.*
>
> **(Stray, 1994, p. 2)**

> *Textbooks are artefacts. They are a part of schooling that many stakeholders have the chance to examine and understand (or*

> *misunderstand). In most classrooms they are the physical tools most intimately connected to teaching and learning. Textbooks are designed to translate the abstractions of curriculum policy into operations that teachers and students can carry out. They are intended as mediators between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms. Their precise mediating role may vary according to the specifics of different nations, educational systems and classrooms. Their great importance is constant.*
>
> **(Valverde, Bianchi, Wolfe, Schmidt and Houang, 2002, p. 2)**

Textbooks play a vitally important role in shaping teachers', students' and families' views of various school subjects (Ravitch, 2003; Valverde *et al.*, 2002). Textbooks themselves reflect particular views of a disciplinary/curricular culture, even if this curricular culture is not necessarily made explicit in the textbooks themselves. Thus, textbooks written in the new or modern mathematics education tradition are likely to differ very significantly in both form and content from textbooks inspired by alternative views of mathematics education, such as Realistic Mathematics Education.

In this section, we draw on the results of the largest ever cross-national study of textbooks, which was undertaken as one of three components in the Third International Mathematics and Science Study (Valverde *et al.*, 2002). The TIMSS curriculum analysis examined mathematics and science textbooks from 48 countries and encompassed a detailed page-by-page inventory of the maths and science content, pedagogy, and other characteristics of hundreds of textbooks in the participating countries for three different age groups. Based on their findings, the authors discuss the rhetorical and pedagogical features in order to understand how textbooks create or constrain students' opportunities to learn complex, problem-solving oriented mathematics or science. In particular, they note the key role that textbooks can play in curriculum-driven educational reform 'in light of their role as promoters of qualitatively distinct educational opportunities' (Valverde *et al.*, 2002). One crucial question Valverde *et al.* (2002) examined was the extent to which textbooks' variability across countries could be linked to cross-national differences in achievement. For the purposes of this report, we concentrate mainly on their findings in relation to mathematics textbooks, focusing on their findings in relation to textbook characteristics, the relationship between textbooks and achievement, and the role of textbooks in translating policy into practice.

Adopting a tri-partite model of curriculum, that is, the intended, implemented and attained curriculum - Valverde *et al.* regard textbooks as mediators of the intended curriculum in that they 'reflect content in their own way - a way that is meant to be suggestive of the enactment of intention. This is done through paradigmatic lessons. Across most TIMSS nations, textbooks are made up of lessons. These lessons are units of instruction written to model valid ways to accomplish the substance of the book' (p. 12).

The TIMSS textbook analysis adopted five measures of textbook characteristics:

- the nature of the pedagogical situation posed by the textbook (e.g. a piece of narration to be read; a set of exercises to be worked on; or an example of how to solve a problem)
- the nature of the subject matter in the textbooks - not in terms of mathematics but rather in terms of topics included, whether they were concrete or abstract, and the degree of complexity of the topics

- sequencing of topics; that is, the number of times attention shifts from one to another
- the physical characteristics of the textbooks; that is, their size, length, weight, etc.
- the complexity of student behaviour 'textbook segments are intended to elicit' (p. 14).

A preliminary analysis, using multivariate analysis of variance (MANOVA), on four hundred textbooks revealed that 'textbooks exhibited statistically significant differences from each other on… characteristics, and that those differences were related to the subject matter of the book; the grade level of the student for which the books were intended; the region of the world in which the textbook was used; and the particular country within each region' (p. 16). Valverde *et al*. strongly emphasise how the variability in textbooks in relation to 'substantial differences in presenting and structuring pedagogical situations' (p. 17) debunks the notion that mathematics has the same connotations in different cultures.

A detailed exposition of the results of textbook characteristics analyses is beyond the scope of this report, but we note key findings. First 'textbooks around the world differ greatly in size, length, and other structural features. They also differ greatly in the types of units they contain and the ways they are laid out' (p. 21). Perhaps the most interesting and thought-provoking finding in relation to textbook size was that it was the sole area in which the USA was number one in the world (Valverde, 1999). Of particular significance was that high-scoring countries on TIMSS (1995) often had compact, focused, light textbooks with in-depth rather than brief focus on topics. In the light of the USA's low rankings in the 1995 TIMSS, the overall findings of the textbook study led US researchers to characterise its textbooks as 'a mile wide and an inch deep' in curricular terms, thereby providing very poor learning opportunities for students. Consequently, US mathematics textbooks were seen as providing a 'splintered vision' of mathematics with serious consequences for how teachers think about mathematics as subject matter, plan and teach lessons, as well as how they think about what it means to know and understand mathematics. Both phrases ('a mile wide and an inch deep' and 'a splintered vision') have come to take on rhetorical power in helping to reshape the form and content of mathematics textbooks and teaching in the intervening years (Cogan, 2005). Internationally, the TIMSS textbook study has provided a context for more careful analysis of how textbooks shape opportunities to learn in mathematics and science. This concern has been given particular impetus since the TIMSS textbook study revealed that, despite bold and ambitious curricular aims of promoting problem-solving and mathematics more focused on the real world, the textbooks analysed in TIMSS (Irish mathematics textbooks were included in two of the three populations) did not live up to these lofty goals. As Valverde *et al*. note:

> *Unfortunately, we have seen here that the textbooks from TIMSS countries are often poor exemplars of such a vision. New subject matter may enter textbooks swiftly but new ways of exploring subject matter in classrooms were largely absent. It is no doubt to be expected that current policies for educational change start out with documenting bold new visions of school mathematics and science. However, it is disquieting that the primary relationship between system-wide policies and the achievement of children is through textbooks. Clearly textbook embodiments of bold new visions in the TIMSS study were very rare. (2002, pp. 170-71)*

The message from the TIMSS textbook study is loud and clear: there is a mismatch in many countries between reform goals in mathematics and the actual mathematics embodied in textbooks. This observation provides a very real challenge for those interested in changing the practice of mathematics education in schools. There have been no studies of post-primary mathematics textbooks in Ireland[4]. However, we note that Looney (2003), in research with teachers working in the support services for post-primary, found that they believed the textbook was more influential than the curriculum in making decisions about classroom teaching. In conclusion, the findings of TIMSS textbook study highlight the manner in which mathematics textbooks and other organised resource materials, function as a potentially implemented curriculum, and thereby help us understand how textbooks act as mediators between intention and implementation. We now turn to tests and examinations which provide another, perhaps more powerful, means of putting in place a vision of mathematics.

Testing and examinations

> *Testing and assessment have been both the focus of controversy and the darling of policymakers…testing and assessment can be externally mandated. It is far easier to mandate testing and assessment requirements at the state or district level than it is to take actions that involve actual change in what happens inside the classroom.*
>
> **(Linn, 2000)**

The use of tests and examinations to select the most suitable for specific positions in society has a long history, going back to tests used to determine entry into the Chinese civil service over four thousand years ago (Zwick, 2002). In the modern era, tests and examinations were given new impetus with the Enlightenment and its faith in

4 Irish mathematics textbooks from 3rd/4th class (primary) and 1st and 2nd year (post-primary) were included in the sample of textbooks in the TIMSS Mathematics Textbook Study.

scientific thinking, rational systems and forms of organisation that could create tests that would more accurately and efficiently determine human abilities and achievements (Broadfoot, 2001). Various testing and measurement technologies, with complex statistical apparatus, have been developed over the last two centuries, and used to identify competence and make fair and just selections in order to allocate people for further training where there were scarce societal resources. Given the perceived and actual close links between test/examination results and future earnings, learners and their families have been acutely tuned to the types of knowledge required to do well in tests and examinations. The post-Enlightenment move away from craft or practical knowledge - highly valued for centuries in trades and crafts - toward the capacity to retain and reproduce knowledge in response to paper-and-pencil tests, has created a situation where retained cognitive knowledge has been elevated over more practical modes of knowledge display (Wells and Claxton, 2002).

One of the major ways in which exam and testing traditions determine curriculum is by shaping what is deemed valuable knowledge. As one Leaving Certificate student, interviewed by a national newspaper after securing high points, said, 'There's no point in knowing about stuff that's not going to come up in the exams.' Of course, if the tests are good in terms of what knowledge they seek to examine and the modes they use to do this, one might argue that this student's strategy is laudable. However, examinations have a powerful backwash effect on curriculum, shaping both what is taught and how it is taught, and often narrow the frame in terms of what counts as worthwhile knowledge. On the other hand, good tests may actually broaden and deepen the quality of what is taught. As such, the actual social and academic effects of tests and examinations may be productive or counterproductive (Mehrens, 1989; Elwood and Carlisle, 2003).

How do the combined effects of curricular cultures, textbooks and exam/test traditions shape how mathematics is taught in schools? In the case of mathematics education at second level in many countries, the impact of the often dominant new mathematics curricular culture and textbooks congruent with this tradition subtly but very powerfully keep in place a view of mathematics that prizes abstraction over concrete experience, formal knowledge over informal knowledge, logical thinking over intuition, reproduction over creative thinking, and routine procedures over problem-solving and problem posing.

1.6 Mathematics education as policy priority

One the major educational features of the current wave of globalisation is a restructuring of the school-society relationship in terms of attention to subject matter content, approaches to teaching, and accountability regarding the outcomes of schooling at school and systemic levels. In the context of setting reform agendas in both primary and post-primary education, globalisation has led to laser-like focus on the knowledge-fields of mathematics and science as they are perceived as providing a greater dividend for, and being more connected to, the marketplace than some other school subjects (Stromquist and Monkman, 2000). Furthermore, Stromquist and Monkman argue that the current wave of globalisation has led to increased attention to pedagogies oriented toward problem-solving and a heightened emphasis on issues of efficiency; note, for example, the growing importance accorded to performance in mathematics and reading tests (Tatto, in press). Across OECD countries, the intense interest in the results of the OECD PISA international comparative assessments of reading, mathematical and scientific literacies is reflected in the burgeoning academic and media publications focused on reporting results, reflecting on the policy implications for education systems, and reframing reform agendas. In the following section, we note the growth of an unprecedented international comparative movement in mathematics education during the last decade, and we highlight developments in post-primary mathematics education within APEC (Asia-Pacific Economic Cooperation) countries (Andrews, 2001a; Andrews, 2001b; Jaworski and Phillips, 1999; Brown, 1999). We then review developments in a number of selected countries: Australia, Japan, Singapore, the United Kingdom and the USA. We adopt a regional or trans-national focus in reviewing policy developments in mathematics education because increasingly countries are banding together at a regional level, in the face of globalisation, in order to calibrate and guide reforms across a wide range of areas, including education, frequently through a peer-review and policy-advisory process. For the purposes of this report, we focus on mathematics education in selected APEC countries for a number of reasons. Firstly, many countries scoring highest on mathematics achievement in TIMSS 1995, TIMSS 1999 and in mathematics literacy in PISA 2000 and PISA 2003 come from this region. Of particular interest in the TIMSS 1995 results was that Japan not only had high aggregate (mean) scores but also had a low level of variation around the mean score. This indicates that overall students, regardless of ability, were taught to a high level. From a policy perspective, Japan's results suggest it had addressed both excellence (high aggregate scores) and equity (little variation about the mean) issues, which is a perennial tension in educational policy and practice (see Table 1). Secondly, significant differences in how high-scoring countries attain high standards in mathematics point to the futility of trying to identify and borrow one or two key factors as reform levers in order to bolster mathematics attainment in any curricular reform efforts. Thirdly, on-going discussion in APEC about educational reform is focusing on how eastern and western education systems might learn

Table 1: TIMSS 1999 video study: participating countries and their average score on TIMSS 1995 and TIMSS 1999 mathematics assessments

	TIMSS 1995 mathematics score		TIMSS 1999 mathematics score	
Country	Average	Standard Error	Average	Standard Error
Australia	519	3.8	525	4.8
Czech Republic	546	4.5	520	4.2
Hong Kong SAR	569	6.1	582	4.3
Japan	581	1.6	579	1.7
Netherlands	529	6.1	540	7.1
Switzerland	534	2.7	-	-
United States	492	4.7	502	4.0
International average	-	-	487	0.7

from each other. In these efforts, policy makers have tried to be sensitive to the varying relationships between national societal priorities and educational practices. This focus within APEC on the need to identify country-specific reform policy instruments, taking account of both a country's existing traditions and its capacity for change, is, we think, valuable and reflects an informed self-review (Le Metais, 2002) approach to educational reform, rather then mere policy borrowing. Finally, given the prominent role the USA plays in setting educational research agendas in mathematics education, reflected in extensive funding over the last decade directed at helping US educators understand how they might learn from the higher-performing East Asian countries, we note some of the issues raised about substantial cross-national differences in 'curriculum policies and materials (such as content coverage, instructional requirements, and structures) but also in how policies and materials are developed in the United States and high performing East Asian countries' (J. Wang and Lin, 2005 p. 3).

1.7 Trans-national alliances in mathematics education policy

'We are all comparativists now'

(EU politician/educator, cited in Phillips, 2003)

One of the distinctive features of contemporary education reform is the focus on international comparative approaches in the curriculum. The development of national education systems has been a long-time focus of international collaboration. For example, a number of key high-level meetings in the late 1950s and early 1960s provided a context for significant increase in investment in education systems in many developed countries, based on a human capital view of education where expenditure came to be seen more as an investment than as a cost for national governments (Husén, Tuijnman and Halls, 1992). More recently, during the last decade, moving beyond what Brown *et al*. (2002) call an older 'walled' economies' approach to economic and educational development, there has been a significant shift toward international collaboration in understanding and improving educational practice in priority curricular areas. Whereas the pioneering work of the IEA, in both the First International Maths Study (FIMS) and Second International Maths Study (SIMS), focused solely on educational achievement, that is, a 'cognitive Olympics' approach, both the Third International Mathematics and Science Study (TIMSS, 1995) and its follow up study TIMSS-R (1999), adopted a much broader remit, focusing also on curriculum textbooks (TIMSS, 1995), a three-country video study (1995), and a seven-country video study (1999). The combined effect of these IEA studies, and the more recent OECD PISA studies (2000 and 2003), has been to unmistakably shift national debates on priority curricular areas into an international arena. As such, over the last decade many countries, either directly or indirectly, have been debating curricular approaches to the teaching and learning of mathematics informed by results of four international comparative studies of mathematics (TIMSS, 1995; TIMSS-R, 1999; PISA, 2000; and PISA, 2003) as well as the compelling insights from both TIMSS 1995 and TIMSS-R 1999 video studies. As many East Asian countries have performed at or near the top of these various international mathematics studies and the 'lesson signature' of Japanese mathematics lessons in the 1995 video study inspired a follow-up video study, we review some of the recent policy trends within the APEC region.

At the third APEC Education Ministerial Meeting in 2004, which focused on 'Skills for the Coming Age', priorities which had been identified over the previous years came together in four thematic strands that reflected the emerging educational needs of the APEC region. 'Stimulation of learning in mathematics and science to promote the technical skills necessary for the 21st century' was one of these four themes. Providing high quality exposure to mathematics and science is viewed as a policy priority within the context of their integration, with overarching educational goals directed at improving students' communication skills, facility with ICTs, and capacity for self-directed learning (APEC, 2004, p. 62). Among the key findings was the manner in which the wider educational governance and societal contexts and values affected approaches to mathematics and science education, as well as attempted curricular reforms in these subjects.

> *Eastern APEC economies tend to do well at promoting and aligning core content knowledge in standards, curricula, assessment and teacher training, but often in inflexible ways that shortchange the needs of individual students and teachers. In Western economies, on the other hand, stress is*

> *placed on individuality and on real world applications of mathematics and science, but core content is not always well taught or understood by all groups of students. (p. 62)*

Both key research findings and reform trends across the region were presented under three headings: (i) curriculum/standards, (ii) assessment and (iii) teachers (see APEC, 2004; Park, 2004).

National initiatives in mathematics education policy

In this section, we discuss post-primary mathematics education in selected countries, focusing on: current policy concerns, current trends in mathematics curriculum, and assessment reform initiatives. In a 16-country study looking at the organisation of mathematics curriculum from ages 4-14 (compulsory schooling), Ruddock (1998) divided countries into two groups based on their educational-political governance structures:

- group A – centralised government: England, France, Hungary, Italy, Japan, Korea, the Netherlands, New Zealand, Singapore, Spain, Sweden

- group B – federal government: Australia, Canada, Germany, Switzerland, USA.

Ruddock noted that, 'There are considerable differences between the two groups in their organisation of mathematics education for ages five to 14, but regional flexibility and differences in local implementation are not unique to federal states.' (1998, p. 2). Ruddock contrasts the emphasis on local implementation in Hungary, Italy, and Spain with Singapore's highly specific guidelines curriculum, Japan's somewhat specific approach through national guidelines, and Sweden's specification of minimum attainment targets. Noting the governance context of Group B countries, she says that 'a national mathematics curriculum as such can only exist provided the regions agree to it both in principle and in practice' (p. 2). In Ireland, there is a national rather than regional approach to curriculum at primary level and syllabus development at post-primary level. Highly decentralised educational governance systems make curriculum reform efforts difficult, especially when it comes to formulating an 'instructional guidance system' to ensure coherence in initiatives intended to improve teaching and learning (Cohen and Spillane, 1992). In this report we can group countries we focus upon into:

- group A – *centralised government:* England, Japan, Singapore,
- group B – *federal government:* Australia, USA

Ruddock notes that the 'Division of the mathematics curriculum into content areas is fairly consistent in the examples available for study, although there are variations in terminology, and not every system includes each content area for a particular age group' (p. 2). She noted the existence of 'basic content building blocks across the curricula studied': number, algebra, geometry, measures, probability, and statistics. She observed that probability was 'absent from several curricula'. In an effort to tabulate the common trends across contexts, she notes that only one element did not fit easily and that was the 'Attitudes and Appreciations section of the Australian National Statement, which is also present in the Singapore framework' (p. 3). Given the importance of promoting an appreciation of mathematics this is a surprising finding. Ruddock emphasises national differences in 'how mathematical process is viewed, sometimes being regarded as a content area, but otherwise seen as cutting across content areas. This aspect of mathematics is absent from the curricula for Korea and Italy… In Singapore, this aspect of mathematics is central to the curriculum framework, but is not used as a content heading in the later primary years' (1998, p. 6). In addition to the specification of curricular content, Ruddock notes that a number of countries also specify performance standards and in some cases include annotated examples of student work at different levels of competence to assist teachers in understanding and implementing the intended curriculum. Ruddock also outlined other efforts being made to assist teachers in understanding reform initiatives. She notes that Singapore, due to its relatively small size relies on meetings with all teachers 'to introduce and implement changes in curriculum, and places particular stress on clarifying the aims of reforms for teachers' (p. 9). Furthermore, the use of assessment is evident in a number of jurisdictions as a means of 'clarifying the meaning of standards [as] is apparent in several systems, for example in Sweden [which has] detailed scoring instructions for tests'. She also drew attention to how national tests in England have shaped teachers' perceptions and 'understanding of the curriculum and their own classroom assessment against performance standards' (p. 10). Ruddock observed that it was difficult to determine the extent to which topics were repeated across years in the various curricula. Given the scale of Ruddock's study, this is not surprising, and it is only with larger and more intensive studies like the TIMSS textbook analyses that the depth and recurrence of particular topics can be traced across time. Furthermore, while Ruddock's study provides a useful set of findings it was not detailed in its analyses of the impact the intended curriculum has on learning, given its focus on curriculum policy documents. As such, it is more distant from the actual enacted curriculum than the TIMSS cross-national textbook study. Nevertheless, Ruddock's study is helpful in providing a curricular context for mathematics education. In the next section, we focus on a selected number of countries, highlighting notable developments in mathematics education.

Mathematical literacy, computing and algebra in Australia

The phrase 'mathematical literacy' is now being used to describe students' capacity to use their mathematical knowledge for informed citizenship. PISA, the new international assessment of 15-year-old students which was conducted under the auspices of the OECD (2001), defines mathematical literacy as an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to engage in mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD PISA, 2001).

The PISA study draws its understanding of mathematical literacy from socio-cultural theories of the structure and use of language. Gee (1996), for example, uses the term *literacy* to refer to the human use of language, the primary function of which is to scaffold the performance of social activities. A literate person knows the resources of the language and can use these for different social functions. In the same way a mathematically literate person knows the resources that mathematics offers and can use these for a variety of purposes in the course of everyday or professional life.

The resources that mathematics offers to a problem-solver include facts, concepts and procedures. Concepts provide the way in which a situation is understood and mathematised so that a problem can be crystallised in mathematical terms, and the procedures are needed to solve the mathematical problem. To harness the power of mathematics, students need to know facts, concepts and skills, the structure of ideas in the domain and how a situation can be mathematised. New technologies are altering the essential understandings of all of these.

Recent work in Australia, and most particularly Victoria, highlights Australia's recent focus on mathematical literacy, especially as it relates to the learning of algebra in the school system with the support of modern technologies. Australia's specific focus has been on the value added by the use of computed algebra systems (CAS): recent work in Australia, and most particularly Victoria, highlights Australia's recent focus on mathematical literacy, especially as it relates to the learning of algebra in the school system with the support of modern technologies. Australia's specific focus has been on the value added by the use of computed algebra systems (CAS) (Leigh-Lancaster, 2002; Leigh-Lancaster, et al, 2003; Stacey, et al, 2000). Their work (sponsored by the government) has focused on changes in the algebra curriculum, in supporting pedagogy and in assessment (classroom and state-wide assessment), with the following goals for mathematical literacy in mind:

- to make students better users of mathematics, by providing the possibility of working on more realistic problems and releasing curriculum time from learning procedures to reallocate to problem solving and modeling
- to increase congruence between mathematics done at school and in the world of work, by using a modern technology
- to achieve deeper learning by students, by using investigations and demonstrations not previously possible

- to promote a less procedural view of mathematics, by shifting curriculum emphasis from routine procedures to solving problems and investigating concepts
- to introduce new topics into the curriculum, using time freed by removing selected routine procedures which are well performed by CAS and have few other pedagogical benefits
- to increase access to mathematics by students with inadequate algebraic skills.

Interestingly, the final goal, increasing access to mathematics by students with inadequate algebraic skills, was one not anticipated by the government, or the researchers, but was by mathematics teachers. These goals are discussed by Stacey, Asp and McCrae (2000), along with the mechanisms by which CAS may help achieve these goals. The development of CAS in Victoria, Australia reflects long-standing collaborative research and policy-making initiatives between academics (e.g. Ball and Stacey, 2005; Stacey, et al, 2000) and the Victorian Curriculum and Assessment Authority (VCAA) (Leigh-Lancaster, et al, 2003). Furthermore, the VCAA also worked closely with other systems and jurisdictions that are using CAS technology in secondary mathematics (e.g. The College Board in the USA, Danish Baccalaureate, French Baccalaureat, IBO) (Leigh-Lancaster, et al, 2003). According to Leigh-Lancaster (personal communication) "the developments in Victoria could only take place within a positive policy orientation towards the use of technology in mathematics curriculum, pedagogy and assessment, including examinations. It is in this regard that the Authority and Board has provided clear and definite leadership and support – the assessment aspect is the hard part – many systems/researchers etc. have been positive about the curriculum and pedagogical inclinations. It takes a robust and progressive conceptualization and resolve to move on the assessment agenda and related issues". The VCAA views the CAS as meeting a wide range of ambitious mathematics teaching goals as well as addressing teachers' pragmatic concerns about the actual teaching of mathematics. According to the VCAA's Mathematics Manager (Leigh-Lancaster, personal communication) these include:

- the possibility for improved teaching of traditional mathematical topics
- opportunities for new selection and organisation of mathematical topics
- access to important mathematical ideas that have previously been too difficult to teach effectively
- as a vehicle for mathematical discovery
- extending the range of examples that can be studied
- as a programming environment ideally suited to mathematics
- emphasising the inter-relationships between different mathematical representations (the technology allows students to explore mathematics using different representations simultaneously)
- long and complex calculations can be carried out by the technology, enabling students to concentrate on the conceptual aspects of mathematics
- the technology provides immediate feedback so that students can independently monitor and verify their ideas

- the need to express mathematical ideas in a form understood by the technology helps students to clarify their mathematical thinking
- situations and problems can be modelled in more complex and realistic ways.

In recent years, "the VCAA has acknowledged and responded to the various reservations and concerns expressed about the use of technology, including by those with a generally positive inclinations, as well as those who are reserved or negative" (Leigh-Lancaster, personal communication). After an extensive consultation on examination models in December 2003 (VCAA, 2003), the VCAA opted for an examination structure that included separate technology free and technology assumed access examinations for 2006 to 2009 (VCAA, 2003).

Post-primary mathematics education in Japan

In Japan, Munbusho, the Japanese Ministry of Education, sets the number of class periods for the year, the length of the class periods, the subjects that must be taught, and the content of each subject for every grade from Kindergarten to 12th grade (K-12). For this reason, changes in the Japanese educational system are usually introduced more cautiously than in, for example, the United States, and possible curriculum revisions are evaluated more carefully before being put into effect. Technology-based courses of the type that one often sees in US classrooms are not as popular in Japan, and Japanese educators generally seem to prefer a more traditional, theoretical, and problem-solving based course. Even though the current curriculum standards encourage the use of calculators beyond the fifth grade, calculators are still not allowed in many Japanese classrooms, since university entrance exams do not permit their use. Computers seem to be more prevalent in the Japanese classroom than hand-held technology.

The elementary school curriculum is specified in Japan for grades 1-6[5]. The objectives of mathematics education at the elementary school level are to develop in children fundamental knowledge and skills with numbers and calculations, quantities and measurements, and basic geometric figures. The emphasis on geometry is far greater than in any of the other OECD countries.

Lower post-primary school in Japan consists of grades 7-9. Preparation to get into the best high schools and universities begins at this time. There is tremendous pressure on students to perform well. Students are asked to learn a tremendous amount of material in grades 7-12, which is perhaps one of the major reasons why university and postprimary school classrooms are often subdued. In

5 In grades 1-3, children learn about the concept of numbers and how to represent them, the basic concepts of measurement, how to observe shapes of concrete objects and how to construct them, and how to arrange data and use mathematical expressions and graphs to express the sizes of quantities and investigate their mathematical relationships. They acquire an understanding of addition, subtraction and multiplication, learn how to do basic calculations up to the multiplication and division of whole numbers, and learn how to apply these calculations. Children also become acquainted with decimal and common fractions during this time. The soroban or abacus is introduced in grade 3. Children learn basic concepts of measurement such as reading a clock, comparing quantities of length, area and volume, and comparing sizes in terms of numbers. They are also taught the concepts of weight and time and shown how to measure fundamental quantities such as length.

By the end of grade 4, children are expected to have mastered the four basic operations with whole numbers and how to electively apply them. They should also be able to do addition and subtraction of decimals and common fractions. In grades 5 and 6, children learn how to multiply and divide decimals and fractions. They are taught to understand the concept of area and how to measure the area of simple geometric figures and the size of an angle, as well as to understand plane and solid geometric figures, symmetry, congruence, and how to measure volumes. Children learn about the metric system during this time. Teachers show how to arrange data and use mathematical expressions and graphs to help children to become able to express the sizes of quantities. Letters such as x and a are introduced. Children also begin to learn about statistical data by using percentages and circle graphs (pie charts). It is recommended that calculators be introduced into the classroom in grade 5 to ease the computational burden.

contrast, elementary classrooms tend to be lively, with a great deal of interaction between students and teachers. In either case, classrooms are teacher-directed. The student-directed group learning that is found in some US classrooms is virtually non-existent in Japan.

In grade 7, students learn about positive and negative numbers, the meaning of equations, letters as symbols, and algebraic expressions. By the end of grade 8, they are able to compute and transform algebraic expressions using letter symbols and to solve linear equalities and simultaneous equations; they have also been introduced to linear functions, simple polynomials, linear inequalities, plane geometry, and scientific notation. In grade 9, students learn how to solve quadratic equations (those with real solutions) and are taught the properties of right triangles and circles, functions, and probability. In grade 7 and beyond it is recommended that calculators and computers should be efficiently used as the occasion demands.

In high school (grades 10-12), six mathematics courses are offered: Mathematics I, II III and Mathematics A, B and C. Although only Mathematics I is required of all students, those students intending to enter a university will usually take all six courses. In fact, Japanese high school students who take all of the courses offered will know more mathematics than many US students do when they graduate from college. In Mathematics I, students are taught quadratic functions, trigonometric ratios, sequences, permutations and combinations, and probability. Mathematics II covers exponential functions, trigonometric functions, analytic geometry (equations of lines and circles), as well as the ideas of limits, derivatives, and the definite integral. Calculus is taught in Mathematics III, including functions and limits, sequences and geometric series, differential and integral calculus. More advanced topics such as Taylor series are not usually taught in Mathematics III. Mathematics A deals with numbers and algebraic expressions, equalities and inequalities, plane geometry, sequences, mathematical induction, and the binomial theorem. Computation and how to use the computer are also taught in this course. In Mathematics B, students learn about vectors in the plane and three-dimensional space, complex numbers and the complex number plane, probability distributions, and algorithms. Mathematics C consists of a variety of topics, including matrix arithmetic (up to 3x3 matrices), systems of linear equations and their representation and solution using matrices, conic sections, parametric representation and polar coordinates, numerical computation including the approximate solution of equations and numerical integration, and some calculus-based statistics.

It is believed that students can attack this rigorous curriculum because of the problem-solving work conducted in the lower grades via the mechanism of 'lesson study' (a programme of teacher-led research on teaching described later in this document). However, the ministry is concerned that the post-primary school system is not producing enough students who complete the six-year mathematics sequence in high school. There is concern in some quarters (again, the ministry) that the post-primary school curriculum is too difficult and stunts student performance over time, especially the creativity developed via lesson study at the elementary school level. This concern has led to very recent meetings between the Ministry of Education in Japan and representatives of the National Science Foundation in the US, with reciprocal goals. The US are interested in understanding lesson study and their Japanese counterparts in how the US system fosters creativity and innovation in the learning of mathematics. We will see below that this search for creativity and the capacity for innovation is also at the forefront of educators' minds in Singapore.

Creativity in mathematics problem solving in Singapore

Being a tiny city-state of four million, Singapore is focused on nurturing every ounce of talent of every single citizen. That is why, although its fourth and eighth class students (roughly equivalent to fourth class and second year students in Ireland) already score at the top of the TIMSS international maths and science tests, Singapore has been introducing more innovations into schools. Its government understands that in a flattening world, where more and more jobs can go anywhere, it's not enough to just stay ahead of its neighbours. It has to stay ahead of everyone.

As Low-Sim Ay Nar, principal of Xinmin Post-primary School, told researchers, Singapore has got rote-learning down cold (Sloane, 2003). No one is going to outdrill her students. What Singapore is now focusing on is how to develop more of America's strength: getting Singaporean students and teachers to be more innovative and creative. 'Numerical skills are very important,' she said, but 'I am now also encouraging my students to be creative - and empowering my teachers… We have been loosening up and allowing people to grow their own ideas.' She added, 'We have shifted the emphasis from content alone to making use of the content,' on the principle that 'knowledge can be created in the classroom and doesn't just have to come from the teacher' (Sloane, 2003).

To that end, some Singapore schools have adopted a maths teaching programme called HeyMath, which was started four years ago in Chennai, India, by two young Indian bankers, Nirmala Sankaran and Harsh Rajan, in partnership with the Millennium Mathematics Project at Cambridge University.

With a team of Indian, British and Chinese mathematics and education specialists, the HeyMath group basically came to the following insight: if you were a parent anywhere in the world and you noticed that Singapore kids, or Indian kids or Chinese kids were doing really well in maths, wouldn't you like to see the best textbooks, teaching and assessment tools, or the lesson plans that they were using to teach fractions to fourth graders or quadratic equations to 10th graders? And wouldn't it be nice if one company then put all these best practices together with animation tools, and delivered them through the internet so any teacher in the world could adopt or adapt them to his or her classroom? That's HeyMath.

'No matter what kind of school their kids go to, parents all over the world are worried that their kids might be missing something,' Mrs Sankaran of HeyMath said in a recent conversation. 'For some it is the right rigour, for some it is creativity. There is no perfect system… What we have tried to do is create a platform for the continuous sharing of the best practices for teaching mathematics concepts. So a teacher might say: "I have a problem teaching congruence to 14-year-olds. What is the method they use in India or Shanghai?".'

According to Mrs Sankaran, Singaporean maths textbooks are very good. They are static and not illustrated or animated. 'HeyMath lessons contain animated visuals that remove the abstraction underlying the concept, provide interactivity for students to understand concepts in a "hands on" manner and make connections to real-life contexts so that learning becomes relevant,' Mrs Sankaran said.

HeyMath's mission is to be the maths Google - to establish a web-based platform that enables every student and teacher to learn from the 'best teacher in the world' for every maths concept and to also be able to benchmark themselves against their peers globally.

The HeyMath platform also includes an online repository of questions, indexed by concept and grade, so teachers can save time in devising homework and tests. Because HeyMath material is accompanied by animated lessons that students can do on their own online, it provides for a lot of self-learning. Indeed, HeyMath, which has been adopted by 35 of Singapore's 165 schools, also provides an online tutor, based in India, to answer questions from students stuck on homework. While Sankaran's comments have the feel and flavour of public relations and sales, it is likely that HeyMath (now with more than a 20% market share) will further catch on in Singapore because of the highly competitive nature of the schooling environment there.

High stakes testing in the US: driving the bull by the tail

In the 1990s, all 50 states in the USA embarked on education initiatives related to high standards and challenging content in mathematics (and science). A central focus of these policy efforts was to establish a common set of academic standards for all, not just elite students. Other components of these standards-based reforms included assessments that measure student performance and accountability systems that are at least partially focused on student outcomes. Bear in mind that graduation from post-primary school has been based solely on classroom teacher-made tests and student performance without comparison across schools at a state or national level. Although assessment has always been a critical component of the education system (Glaser and Silver, 1994; Linn, 2000), the growing focus on standards (for example the NCTM standards documents, 2000) and accountability has dramatically changed the role of tests in the lives of students, their teachers and their schools. Teachers continue to use the results of classroom and other types of tests to plan instruction, guide student learning, calculate grades, and place students in special programmes. However, with the passing of President George Bush's No Child Left Behind (NCLB) Act of 2001, policy makers are turning to data from large-scale state-wide assessments to make certification decisions about individual students, and to hold teachers accountable for their performance and progress of their students.

Provisions in the federal government's Title I programme have reinforced the role of assessment in standards-based reform. Title I of the Improving America's Schools Act (IASA) of 1994 required states to develop high-quality tests aligned with state standards in reading and mathematics in one grade per grade-span (elementary, middle and high school), and to use these data to track student performance and identify low-performing schools (somewhat akin to the league tables in use in Great Britain). The most recent amendments to Title I, contained in the NCLB Act of 2001, give even greater prominence to state assessment. The law expanded state testing requirements to include every child in grades 3 to 8 in reading and mathematics by the 2005-06 school year, and in science by 2007-08. These assessments must be aligned with each state's standards and allow for student achievement to be comparable from year to year. The results of the tests will be the primary measure of student progress toward achievement of state standards. States will hold schools and districts accountable for 'adequate yearly progress' toward the goal of having all students meet their state-defined 'proficient' levels by the end of the school year 2013-14. Students attending Title I schools that fail to make adequate progress are given the option of transferring to other public schools or receiving supplemental educational services outside of school. Title I schools that fail to improve over time can be restructured, converted to charter schools, or taken over by their district or state.

United Kingdom: making maths count and counting maths teachers

One of the big concerns in England, as against the more favourable situation in Scotland, is the lack of suitably qualified teachers to teach mathematics at second level. Furthermore as maths appears to count more and more in economic and scientific progress, counting (and reversing) the declining numbers of mathematics teachers is a key policy focus. These concerns are reflected in a recent report, *Making Mathematics Count: The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education* (2004). The genesis of the report was partly due to an earlier report on science, engineering and technology (the Roberts Report: SET for Success: the supply of people with science, technology, engineering and mathematical skills published in 2002) which according to Smith (2004) noted that, although

> *…relative to many other countries, the UK has a large and growing number of young people studying science and engineering, this overall growth has masked a decline in the numbers studying the physical sciences, engineering and mathematics. For example, the report drew attention to the drop during the 1990s of nearly 10 per cent in the numbers taking A-level mathematics in England. At the same time, the report also noted that the demand for graduates and postgraduates in these strongly mathematically oriented subjects has grown significantly over the*

> *past decade, not only in science and engineering areas, but also in the financial services and ICT sectors. In addition to the supply problem, the report identified concerns expressed by employers about the mismatch between skills acquired during formal education and those required in the workplace.' (2004, p. 1, Section 0.3)*

Smith went on to note that the mismatch problem identified in the Roberts Report had potentially serious consequences for the economy because it 'is leading to skills shortages that will adversely affect the government's productivity and innovation strategy. These shortages will become increasingly serious unless remedial action is taken' (2004, p. 1, Section 0.3). The terms of reference for the Smith Report, commissioned by the UK government were as follows: to make recommendations on changes to the curriculum, qualifications and pedagogy for those aged 14 and over in schools, colleges and higher education institutions to enable those students to acquire the mathematical knowledge and skills necessary to meet the requirements of employers, and of further and higher education. (2004, p. 2, Section 0.8). Indicative of the ongoing concern about mathematics education in the United Kingdom is the appointment of Celia Hoyles as mathematics 'czar' for the UK government. This appointment reflects a rising concern about spiralling standards in mathematics education. A recent press release from the Department of Education and Skills typifies the politically sensitive nature of mathematics education:

> *No government has done more to get the basics right in our schools….Standards in maths are rising with last summer's test and exam results showing good progress. In 2004 maths had the the highest entry rate of any GCSE and the third highest entry rate at A-level…. Vacancies for maths teachers have declined every year since 2001 and to boost recruitment further, graduates from 2006 will be offered a £9,000*

bursary and £5,000 'golden hello' to train as maths teachers. (28th June 2005, BBC News, www.bbc.co.uk)

Finally, the UK government is about to establish a national centre for excellence in the teaching of mathematics.

Ireland: slow emergence of concern about mathematics education

The PISA shock that led to national concern in Germany about mathematics or the 'splintered vision' concerns about mathematics education in the USA have not been mirrored in Ireland, despite the fact that on international tests Irish students perform at a moderate level (mid-ranking in both PISA 2000 and 2003 in mathematics literacy). The less dramatic reaction in Ireland may be wise or unwise depending on your perspective. Given that curriculum reforms, if undertaken, are best done in a considered rather than reactionary manner, less immediate concern may be justified. On the other hand, bearing in mind the ambitious social and economic goals being set in the fast-growing economy, it could be argued that a more concerted effort to address perceived weaknesses in mathematics education is merited. A series of studies over the last three years have drawn attention to the state of mathematics education in Ireland. These include the landmark video study *Inside Classrooms* (Lyons *et al*. 2003) which provided a rich portrait of mathematics education at junior cycle (lower post-primary). Recent national reports on PISA 2000 (published in 2003) and PISA 2003 (published in 2005), as well as two curriculum mapping exercises (one mapping current mathematics syllabi onto PISA and the other mapping PISA onto examination items), are providing a variety of very useful data in promoting debate about the state and vision of mathematics education in Irish post-primary classrooms. Among the insights emerging from these recent research studies are the following:

- post-primary mathematics education in Ireland is taught in a highly didactic and procedural fashion with relatively little emphasis on problem solving (Lyons, *et al*., 2003)
- the 'new' mathematics curricular culture, with its elevation of abstraction as its core principle, has dominated post-primary mathematics teaching for the last forty years (Oldham, 2001)
- in both PISA 2000 and 2003, Ireland is ranked in the middle of OECD countries in mathematical literacy (Cosgrove, *et al*., 2005).

1.8 Conclusion

This chapter has provided an overview of key issues in mathematics education internationally. We address many of the issues raised in Chapter 1 in greater detail in the rest of this report. Among the key issues raised in this chapter were

- the existence of multiple goals in debates on mathematics education. For example, the 'mathematics for scientific advancement' and the 'mathematics for all' goals
- the mathematical sciences as changing and dynamic areas of knowledge
- the changing definition of mathematics, that is, from 'working with numbers' to the 'science of patterns' (see Table 4, Chapter 3)
- the high level of policy priority accorded to mathematics by many countries, since it is seen as a high-yield subject in preparing workers and citizens for the knowledge society

- the combination of the push factor (that is, poor levels of understanding and achievement gaps related to gender, SES, ethnicity), and the pull factor (that is, the demands of 21st century economy and society), which are fuelling concern about mathematics education internationally

- mathematics education as a contested terrain: on the one hand, the formal and abstract view of maths in new or modern mathematics which has dominated since the 1960s; on the other, a more 'real world', problem-focused mathematics underpinned by cognitive science and realistic mathematics education

- the differing meanings of 'basic' in debates about mathematics

- three factors that help us understand why mathematics is taught the way it is: curricular cultures, textbooks, and testing/ examination traditions

- the mismatch between ambitious problem-solving oriented curriculum goals in mathematics education compared to the procedural approach and content of mathematics textbooks

- the important role played by educational governance structures in framing curriculum reform initiatives

- the research finding that post-primary mathematics education in Ireland is typically taught in a highly didactic and procedural fashion, reflecting the impact of the 'new' or 'modern' mathematics movement teaching over the last forty years

- recent developments that have helped to inform new visions of mathematics education: video and other technologies to understand teaching practices; new images of learning; new goals for education/schooling.

As video studies of post-primary mathematics teaching have had a major impact internationally on promoting an understanding of classroom practices in mathematics education, the next chapter focuses on key findings from video studies undertaken in the last decade.

CHAPTER 2:

Understanding and improving teaching: video studies and lesson study

2.1 Introduction

This chapter focuses on developments in understanding the teaching of mathematics, drawing primarily on compelling findings from the TIMSS 1995 and TIMSS-R 1999 video studies. As one of the key insights of both video studies was the nature of culturespecific 'lesson signatures', the second section of the chapter provides a review of developments in Japan and elsewhere in relation to *Lesson Study* as a form of teacher professional development and curricular reform. We think the focus on lessons learned both from these landmark video studies and Japanese Lesson Study reflects what Ball *et al.*, writing in the fourth *Handbook of Research on Teaching* (2001), identifed as a move *from teachers to teaching* as the focus of research on teaching. This shift in focus towards actual classroom practice is also found in research on mathematics education. For example, because of the impact of the cognitive revolution, research on teaching since the 1970s had focused on teacher thinking (Clark and Peterson, 1986) and knowledge: how do teachers think and what do they need to know to teach high-quality mathematics curriculum (Munby, Russell and Martin, 2001). This research has been complemented over the last decade by a shift towards carefully examining the actual practice of mathematics teaching through structured observation, case studies (sometimes supported by video data), video surveys, or video-ethnographies.

2.2 Understanding classroom practice: The critical role of video

Why use video as a form of data in the study of classroom instruction? Traditionally, attempts to measure classroom teaching on a large scale have used teacher surveys (i.e. questionnaires). In general, questionnaires are economical and relatively easy to administer to a large number of respondents, and responses usually can be transformed readily into data files ready for statistical analysis. However, using questionnaires to study classroom practices can be problematic because it is difficult for teachers to remember classroom events and interactions that occur quickly. Moreover, different questions can mean different things to different teachers, thereby creating validity issues (Stigler, 1999).

Direct observation of classrooms by coders overcomes some of the limitations of questionnaires, but important ones remain. Significant training problems arise when used across large samples, especially in cross-cultural comparisons. A great deal of effort is required to ensure that different observers are recording behaviours in comparable ways. In addition, as with questionnaires, the features of the teaching to be investigated have to be decided ahead of time. While new categories might occur to the observers as the study progresses, the earlier classes cannot be re-observed.

Video offers a promising alternative for studying teaching (Stigler, Gallimore and Hiebert, 2000), with special advantages. For example, by preserving classroom activity, video enables detailed examination of complex activities from different points of view. However, video is not without its disadvantages (see Jacobs *et al.*, in press); these are discussed later in this monograph. For the moment we focus on a component part of video that is rarely discussed, that of sampling.

2.3 Many types of video studies: From video surveys to video cases

Rapid advances in video technology, more user-friendly video technology, and declining costs have made it possible for relatively small-scale research projects to make effective use of video as a data-collection tool in research on classroom practices (Reusser, Pauli, and Hugener, 2005). Following Stigler *et al.* (1999), Roth (2004), and Reusser *et al.* (2005), we can summarise the advantages and affordances of video as follows:

- It allows research questions and analytic categories to emerge from the data: 'A "record" of actual events that can be observed repeatedly' (Reusser, Pauli and Hugener, 2005).
- It provides raw uninterpreted data.
- It is less theory bound than other methods of data collection.
- It creates the possibility of interdisciplinary and intercultural meaning negotiation.
- It enables repeated analyses of the same data from different theoretical angles at a later time.
- It provides 'concrete referents to find a shared language about teaching processes' (Reusser *et al*, 2005).
- It facilitates the integration of qualitative and quantitative analyses, and thus helps to transcend the dichotomy between qualitative and quantitative research.
- It creates the possibility 'for more nuanced analyses of complex processes that are tied to observable evidence' (Reusser *et al.*, 2005).

Over the last decade a variety of video studies have been undertaken in mathematics education (see Table 2). While there are numerous issues of interest among the different foci, settings and findings of

Table 2: Video studies in mathematics education

Name + study type	Sample	Lessons: number and content focus	Key insight or question
TIMSS 1995 Video survey	3-country study: nationally representative sample of 8th grade maths lessons in USA, Germany and Japan.	231 lessons; one camera per classroom.	Is there such a phenomenon as 'cultural script' to describe lessons at national level?
TIMSS-R 1999 Video survey	7-country study: nationally representative sample of 8th grade maths lessons in Australia, the Czech Republic, Hong Kong SAR, Japan, the Netherlands, Switzerland and the United States.	638 lessons (including 1995 Japanese lessons); one camera per classroom.	High quality mathematics teaching takes many different forms.
Learner's Perspective Study (Clarke, 2002) Video survey	9-country study at 8th grade: Australia, Germany, Hong Kong, Israel, Japan, the Philippines, South Africa, Sweden and the USA.	Focus on sequence of 10 or more 'well-taught' lessons; three cameras per lesson: whole class, teacher and student.	Video-stimulated reconstructive recall of lessons is valuable in understanding student and teacher intentions.
Ms Ball (Ball, 1993) Video self-study	Self-study over one year at 4th grade, East Lansing, MI, USA.	Most lessons over the course of one school year were video-taped.	Student sense making provides many rich opportunities to foster important mathematical ideas.
Ms Smith (Cobb et al., 1997) Video case	Case study of one 1st grade teacher over a year in Nashville, TN, USA.	106 lessons; two cameras.	An example of integrating social and psychological frameworks in research on maths teaching.
Inside Classrooms Lyons et al., 2001) Video case	Junior cycle mathematics in Ireland at 9th grade.	20 lessons from 10 schools one camera per classroom.	Striking homogeneity across lessons/schools.

these studies, the somewhat different sampling of lessons is an important difference between them. The TIMSS studies focused on one lesson per teacher, whereas the Learner's Perspective Study, assuming that teaching involves thoughtful sequencing of content congruent with learners' developing understandings, focused on sampling a sequence of 'well taught' lessons (Clarke, 2002). For the purposes of this report, we focus on the TIMSS video studies as they provide a range of important insights into the rhythms of mathematics teaching and conditions for changing classroom practices.

TIMSS video studies

The broad purpose of the 1998-2000 Third International Mathematics and Science Study Video Study (TIMSS 1999 Video Study) was to investigate and describe teaching practices in Year 8 (i.e. second/third year mathematics in Ireland) mathematics and science in a variety of countries. It is a supplement to the TIMSS 1999 student assessment, and a successor to the TIMSS 1995. The TIMSS 1999 video study expanded on the earlier 1994-95 video study (Stigler *et al.*, 1999) by investigating teaching in science as well as mathematics, and sampling lessons from more countries than the original study.

TIMSS 1995 Video Study included only one country, Japan, with a relatively high score in Year 8 mathematics as measured by the TIMSS assessment instruments. It was tempting for some audiences to prematurely conclude that high mathematics achievement is possible only by adopting teaching practices like those observed in Japan. In contrast, the TIMSS 1999 Video Study addressed these concerns by sampling Year 8 mathematics lessons in more countries - both Asian and non-Asian, where students performed well on the TIMSS 1995 mathematics assessment. Countries participating in the TIMSS 1999

Video Study were Australia, the Czech Republic, Hong Kong SAR[6], Japan, the Netherlands, Switzerland, and the United States.

The aim of the TIMSS 1999 Video Study

The video survey methodology used in the TIMSS 1999 Video Study enabled very detailed snapshots of mathematics teaching to be collected. Internationally, a general aim was to use these snapshots to describe patterns of teaching practice in the participating countries. More specific aims included

- development of objective, observational measures of classroom instruction to serve as quantitative indicators of teaching practices
- comparison of teaching practices to identify similar or different features across countries
- development of methods for reporting results of the study, including preparation of video cases for both research and professional development purposes.

Scope of the study

The mathematics component of the TIMSS 1999 Video Study comprised 638 Year 8 lessons collected from all seven participating countries. The 50 lessons collected in Japan in the 1995 video study are included in this tally. The required sample size in 1999 was 100 lessons per country. One lesson per school was randomly selected from each of the approximately 100 randomly selected schools per country.[7]

6 It is this document Hong King SAR is referred to as a country. Technically, Hong Kong is a special administrative region (SAR) of the People's Republic of China.

7 The weighted response rate reached the desired 85 per cent or more in all countries except the United States, where it was 76 per cent.

In each school the selected teacher was filmed for one complete Year 8 mathematics lesson, and in each country (except Japan) videotapes were collected throughout the school year to try to capture the range of topics and activities that can occur across a whole school year. To obtain reliable comparisons among countries the data were appropriately weighted according to the sampling design.

Processing the data was a long, complex and labour-intensive undertaking. Several specialist teams were needed to decide what should be coded, what kinds of codes to use, and how reliably the codes could be applied. Many revisions were made to the codes before a satisfactorily reliable and common set were put in place. All coding was completed under the guidance of James Stigler at LessonLab on the campus of UCLA. Country representatives were encouraged to reside in Los Angeles for the period of coding.

Major findings

In general terms, the TIMSS 1999 Video Study of Year 8 mathematics teaching showed that there is no one way to undertake successful teaching of mathematics, bearing in mind that the countries studied were chosen because, other than the US, their students had performed well in the TIMSS assessments and they were willing to participate (see Table 3 below).

Table 3: Average scores on TIMSS 1995 and TIMSS 1999 mathematics assessments

Country	1995	1999
Australia	519	525
Czech Republic	546	520
Hong Kong SAR	569	582
Japan	581	579
Netherlands	529	540
Switzerland	534	-
United States	492	502

The results showed that teachers in these high performing countries used a variety of teaching methods and combined them in different ways. All countries shared come common features, while most countries were found to have some distinctive features.

Common features included

- At least 95 per cent of lesson time, on average, was spent on mathematical work (excluding time taken to organise students in relation to these tasks).

- At least 80 per cent of lesson time, on average, was spent solving mathematical problems, regardless of whether the main purpose of the lesson was review of previously explored content or the presentation or practice of new content.

- Lessons generally included some review of previous content as well as some attention to new content.

- Most of the time, lessons included some public, whole-class work and some private, individual or small group work - during the private time, students mostly worked individually rather than in pairs or larger groups.

- At least 90 per cent of lessons made use of a textbook or worksheet of some kind.

- Teachers talked much more than students, both in terms of numbers of words uttered and in terms of the lengths of such utterances. The ratio of teacher to student words was 8:1. Most teacher utterances were at least five words long while most student utterances consisted of fewer than five words.

Distinctive features

These features were found in relation to the introduction of new content, the emphasis on review of previous content, the use of various strategies to make lessons more coherent, the mathematical topics covered, the procedural complexity of the problems discussed and assigned, and the classroom practices regarding the use of individual work time and the use of class time for homework. Findings on these and other variables can be found in Hiebert *et al., Teaching Mathematics in Seven Countries* (2003).

2.4 Understanding classroom practice: Japanese lesson study

In this section we describe what has in the last 20 years become known as 'lesson study', a central component of Japan's major effort at teacher professional development (*kounaikenshuu*) in elementary and middle school teaching (infant through second year teaching in Irish schools). We begin by outlining the processes associated with lesson study and how they are developed and utilised. Next we describe the meta-context for education and schooling in Japan. We note that, given the number of students attending evening cram schools in Japan, Japan's international performance in mathematics draws on the combined effects of lesson study and *juku* (Japan's cram school system). We then highlight a finding from the US's National Academy of Science report (see Snow, Burns, and Griffin, 1998) on the current debate in reading circles about phonics versus whole-language learning. The Academy notes that reading with competence and comprehension is probably due to both strategies. We suggest that this is likely to be the case for the learning of mathematics. However, this should not detract from the fact that the quality of instruction found in Japanese classrooms is something that should be emulated where possible world-wide. In an effort to better link research and practice we ask if the process of lesson study could be improved by drawing on current research knowledge (in the first iterate of the lesson study process).

In their now classic book, *The Teaching Gap* (1999), Stigler and Hiebert contrast mathematics instruction in three countries: Germany, Japan and the United States. The authors draw on randomly sampled lessons from the TIMSS 1995 video study of eight grade mathematics instruction. The countries were picked for a number of reasons: professional, financial and political. Japan was chosen because of its high performance in SIMS (the Second International Maths Study) and expected high performance in TIMSS, while Germany was selected because of anticipated changes in performance due to the merging of East with West Germany in the period between SIMS and TIMMS. The US wanted to participate because it was felt that a lot could be learned through the contrasting of the three systems and because the US was underwriting a large proportion of the study.

Stigler and Hiebert (1999) use their video analysis to highlight instructional differences across the participating countries, asking why teachers in each country teach differently and what it is about the educational systems of each country that supports the different types of instruction that can be seen from nation to nation. Additionally, they outline the policy issues raised by their work for the teaching and learning of mathematics in American schools. It is in this regard that they provide insight into the processes underlying lesson study in Japan.

In lesson study, groups of teachers meet regularly over long periods of time (ranging from several months to a school year) to work on the design, implementation, testing and improvement of one or more 'research lessons'. Lesson study is a deceptively simple idea. If you want to improve instruction, what could be more obvious than getting together with your peers to collaborate on the development and implementation of lessons? While a simple idea, lesson study is quite complex in practice, with many component parts. Stigler and Hiebert (1999) focus our attention on what they believe to be the eight common parts of lesson study. These include: defining the problem; planning the lesson; teaching the research lesson; evaluating the lesson and reflecting on its effect; revising the lesson; teaching the revised lesson; evaluating and reflecting again; and sharing the results. Lewis (2002), on the other hand emphasises the metastructure in what she labels the lesson study cycle. She attends to four critical structures: goal setting and planning; researching the lesson; lesson discussion; and consolidation of learning. In the discussion that follows we combine the two renderings as appropriate. While we present the general features of lesson study we include, where appropriate, what teachers are likely to do. For more detail see Lewis (2002) and Fernandez (2004). Lewis provides rich examples of lesson study for the learning of primary school science, whereas Fernandez explicitly attends to the learning of mathematics. A discussion of lesson study in the Irish context is available in Kelly and Sloane's recent article in *Irish Educational Studies* (2002).

Goal setting and planning

1. Defining the problem. At its essence lesson study is the act of problem-solving (problem-solving by teachers about teaching and instruction). Consequently, the first step in the process is to define the problem to be resolved. What is the central problem that motivates the work of the lesson study group? Here the teachers identify the goals for student learning and for long-term development. These goals can be general or specific in nature,

focusing on student motivation or on the learning of a specific mathematical topic. Normally, teachers pick a topic that arises from their daily practice. Occasionally, topics come from the Ministry of Education. When this arises they are of the form 'How can we help students learn x?' The Ministry then invites a sample of schools throughout the country to look at this question using lesson study as a vehicle for generating this knowledge. The schools are then expected to report back on their findings. Additionally, the Ministry will issue recommendations in a top-down fashion. The interplay between the bottom-up and top-down mechanisms for change is unique to Japan. This feature of educational policy allows teachers direct input into national policy in a manner not available in other countries. This is not a common policy feature in Ireland.

2. Planning the lesson. When the learning goal has been set teachers meet regularly to plan the lesson. The aim here is twofold: to produce an effective lesson and also to understand why and how the lesson promotes (or doesn't promote) improved mathematical understanding on the part of students. Often teachers begin the planning process by looking at books and articles produced by other teachers who have studied a somewhat similar problem. This initial planning process can occur over weekly meetings for a period of months. In general, the lesson begins with the statement of a mathematical problem. Consequently, teachers engage in detailing the following:

    a. the mathematical problem, its exact wording, and the numbers to be used

    b. the materials students would need to be given to address and solve the problem

c. how the materials are likely to be used by the students

d. the possible solutions that students are likely to generate as they struggle with the problem; the probable misconceptions

e. the questions the teacher will use to promote student mathematical thinking during the lesson

f. the types of guidance that teachers could provide to students who showed one or another misconceptions in their thinking

g. the use of the blackboard by the teacher

h. the allocation of the finite amount of time devoted to the lesson and its component parts (the introduction of the problem, student work, etc.); how to handle individual differences in the level of mathematical preparedness of the students in the class

j. how to close the lesson. Lesson endings are considered crucial and involve teachable moments to advance student understanding.

The research lesson

3. Teaching the lesson. The team (that is, all teachers) prepare the lesson together with role-playing. One teacher teaches the lesson. The others attend the lesson with specific tasks in mind. When the students are asked to work at their desks, the other team members collect data on student thinking, what is being learned, student engagement, and student behaviour. They do this by observing and taking careful notes. Occasionally, the lesson is video-taped for further discussion.

Lesson discussion

Here the teachers share and analyse the data they have collected at the research lesson. They carefully examine evidence as to whether the goals of the lesson for student learning and development were fostered (or met). Finally they consider what improvements to the lesson specifically, and to instruction more generally, are necessary.

4. Evaluating the lesson and reflecting on its effect. On the day the lesson is taught the teachers will meet to discuss what they observed. In general the teacher who taught the lesson speaks first generating their own personal evaluation. Then the other teachers contribute. Comments tend to be quite critical in nature, but focus on the group product, the lesson itself, and not on the teacher. All team members feel responsible for the final group product. This is an important activity because the team places attention on the possibility and opportunity for lesson improvement.

5. Revising the lesson. Next the teachers revise the lesson as a group. They do this based on their observational data and on shared reflections. They may change any number of things in the lesson, including the materials, the specific activities, the opening problems posed, the questions asked, or on occasion all of these components. The teachers base their changes on misunderstandings demonstrated by the students they have observed.

6. Teaching the revised lesson. When the revised lesson is formulated it is then taught to a new classroom of students. It may be taught by the first teacher but it is often, and more than likely, taught by another team member. It is the lesson that takes central focus, not

the teacher. During this iterate other teachers may be invited to observe. Often the full complement of teaching staff is invited to the second or third iterate of the lesson. The goal is simple; it is one of slow, deliberate, iterative revision in search of perfection.

7. Evaluating and reflecting. Again, the teacher who taught the lesson comments first. The teacher will describe the goals of the lesson, his or her own assessment of how the lesson went and what parts of the lesson the teacher feels need re-thinking. Observers then critique the lesson and suggest changes based on their observations. An outside expert may be invited to these deliberations. The lesson conversations vary in focus from the specific issues of student learning and understanding, but also with respect to more general, or meta-issues, raised by the initial hypotheses used to guide the design and implementation of the lesson.

8. Consolidation of learning. If desired, the teachers refine and re-teach the lesson and study it again. When the iterative process is complete they write a report that includes the lesson plan, the student data, and the reflections on what was learned by the students, and by the teachers about the process of learning. This description has focused on a single lesson, but Japan, like Ireland and unlike the US for example, has a well-articulated national curriculum. Consequently, what the teachers have learned has relevance for other teachers locally and nationally. Teachers (and the school system) need to embrace the opportunity to share results. This can be undertaken in a number of ways. We highlight the ways that learning is consolidated locally and more globally in Japanese schools. First, as noted above, teachers write a report. The report documents the lesson, the reason for the lesson, what is

learned by students and by teachers, what the original assumptions were, and how these assumptions changed through the process of teaching, observation and lesson revision. The report is bound and is available to all teachers in the local school. It is read by faculty (consider the power of this resource for the new or neophyte teacher); it is read by the principal; it is shared within the prefecture if considered interesting enough. If a university professor collaborated with the group the work may find an even broader audience. Another way of consolidating and sharing the lesson study results occurs when teachers from other schools are invited to a rendering of the final version of the lesson. Sometimes, 'lesson fairs' are conducted at schools and teachers from a neighbouring geographic region are invited to watch research lessons in many subjects produced by a school over an extended time period. These are considered festive occasions, and a critical part of ongoing teacher development.

Some reflections

Lesson study is of course culturally based, and we discuss the culture of Japanese schooling below. We believe that the process of lesson revision is a powerful one and one that can be legitimately adopted and adapted to fit the Irish school system. Lesson study will not bring quick rewards but it is a deliberate and concentrated effort to improve the process and products of teaching in mathematics.

Lesson study re-situated: the wider context of schooling in Japan

Understanding the Japanese people and culture requires understanding the factors that mould them. Particularly important are those components which influence them in their formative years, notably the education system. Given the large amount of time that Japanese students spend in schools, it is little wonder that the education system plays a tremendous role in determining the fabric of Japanese society. An examination of the 'typical' post-primary school experience illuminates the function of the education system in Japanese society. One of the lessons from recent research in educational anthropology and cultural psychology is the importance of understanding educational activities and academic performance in their wider social and cultural contexts. Thus, in order to provide a context for the interpretation of Japanese post-primary mathematics education, we provide a brief description of students' school, extra-curricular, cram school and entrance examination experiences.

At school

Japanese students spend 240 days a year at school, approximately 60 days more then their Irish counterparts. Although many of those days are spent preparing for annual school festivals and events such as culture day, sports day and school excursions, Japanese students still spend considerably more time in class than their Irish counterparts. Traditionally, Japanese students attended school for half a day on Saturdays; however, the number of required Saturdays each month is decreasing as the result of Japanese educational reforms. Course selection and textbooks are determined by the Japanese Ministry of Education. Schools have limited autonomy in their curriculum development. Students in academic high schools typically take three years of each of the following subjects: mathematics, social studies, Japanese, science and English. Other subjects include physical education, music, art and moral studies. All the students in one grade level study the same subjects. Given the number of required subjects, electives are few.

At the end of the academic day, all students participate in *o soji*, the cleaning of the school. They sweep the classrooms and the hallways, empty rubbish bins, clean bathrooms, clean chalkboards and chalk dusters, and pick up litter from the school grounds. After *o soji*, school is dismissed and most students disperse to different parts of the school for club meetings.

Extracurricular activities

Club activities take place after school every day. Teachers are assigned as sponsors, but often the students themselves determine the club's daily activities. Students can join only one club, and they rarely change clubs from year to year. In most schools, clubs can be divided into two types: sports clubs (baseball, soccer, *judo, kendo*, track, tennis, swimming, softball, volleyball, rugby) and culture clubs (English, broadcasting, calligraphy, science, mathematics, yearbook). New students are usually encouraged to select a club shortly after the school year begins in April. Clubs meet for two hours after school each day and many clubs continue to meet during school vacations. Club activities provide one of the primary opportunities for peer group socialisation.

Most college-bound students withdraw from club activities during their senior year to devote more time to preparation for university entrance examinations. Although visible in the general high school experience, it is in the clubs that the fundamental relationships of *senpai* (senior) and *kohai* (junior) are established most solidly. It is the responsibility of the *senpai* to teach, initiate and take care of the *kohai*. It is the duty of the *kohai* to serve and defer to the *senpai*. For example, *kohai* students in the tennis club might spend one year chasing tennis balls while the upperclassmen practise. Only after the upperclassmen have finished may the underclassmen use the courts.

The *kohai* are expected to serve their *senpai* and to learn from them by observing and modelling their behaviour. This fundamental relationship can be seen throughout Japanese society, in business, politics, and social dealings.

Juku: cram schools

An interesting component of Japanese education is the thriving industry of *juku* and *yobiko*, after-school 'cram schools', where a large proportion of Japanese high school students go for supplemental lessons. *Juku* may offer lessons in non-academic subjects such as art, swimming, abacus and calligraphy, especially for elementary school students, as well as the academic subjects that are important to preparation for entrance examinations at all levels. *Juku* for high school students must compete for enrolment with *yobiko*, which exist solely to prepare students for university entrance examinations. Some cram schools specialise in preparing students for the examination of a particular school or university. Although it would seem natural for students to dread the rigour of additional lessons that extend their school day well into the late evening hours and require additional homework, many students enjoy *juku* and *yobiko*, where teachers are often more animated and more interesting than some of the teachers in their regular schools. Remember that the role of the teacher as portrayed by Stigler and Hiebert (1999) is to draw the mathematics out from the student. This requires that the teacher listen to the student more than is typical in Irish classrooms.

Juku and *yobiko* are primarily private, profit-making schools that attract students from a wide geographical area. They often are located near train stations, enabling students to transport themselves easily to *juku* directly from school. *Juku* and *yobiko* thrive in Japan, where it is believed that all people possess the same innate intellectual capacity, and it is only the effort of individuals, or lack thereof, that determines their achievement above or below their fellows. Much like Ireland, in Japanese schools there is the tendency to pass students up to the next 'year' with their entering cohort. Therefore, without the supplemental *juku* lessons, some students could fall well behind their classmates. *Yobiko* also exist to serve *ronin*, 'masterless samurai', students who have failed an entrance examination, but who want to try again. It is possible for students to spend a year or two as *ronin* after graduating from high school, studying at *yobiko* until they can pass a university entrance examination or until they give up. Cram school tuition is expensive, but most parents are eager to pay in order to ensure acceptance into a selective junior high school, high school or university, and thus a good future for their children.

Entrance examinations

In addition to university admission, entrance to high school is also determined by examination, and the subjects tested are Japanese, mathematics, science, social studies and English. Private high schools create their own examinations, while those for public high schools are standardised within each prefecture. Students (and their parents) consider each school's college placement record when deciding which examinations to take. Success or failure in an entrance examination can influence a student's entire future, since the prospect of finding a good job depends on the school attended. Thus, students experience the pressure of this examination system at a relatively early age. The practice of tests at school and *juku* helps teachers to direct students toward institutions whose examinations they are most likely to pass.

What do lesson study and the context of learning in Japan mean for mathematics teaching and learning in Ireland?

Le Métais (2003) warns policy makers that they should be careful when they simply copy one system of education with the vain hopes that such a system will work in the context of another culture. To that end we raise a number of salient questions to help us make local policy for Ireland that draws on cross-national educational study. To summarise Le Métais (2003), sensible policy making results from cross-national study when it helps us to (1) build informed self-review, and (2) clarify the goals of mathematics education.

Japanese performance in international comparisons is based not only on high quality lesson study but also on a system of cram schools. This should alert us to the dangers of drawing simple conclusions from the Japanese experience. Firstly, it is difficult if not impossible to disentangle the possible confounding effects of this dual system of education. For example, there is the anomaly that no homework is assigned in primary and the early years of secondary schooling. Stigler and Hiebert (1999, p. 30) note that 'no homework is typical in Japan' in their study of 8th grade. This is sometimes regarded as a weakness. As recently as 25 July 2005, Minako Sato wrote in the Japan Times: 'Cram schools cash in on the failure of public schools.' Further, she notes that, 'according to a 2002 survey by the Ministry of Education, Culture, Sports, Science and Technology, 39 percent of public elementary school students, 75 percent of public middle school students, and 38 percent of public high school students attend juku.' While one part of the system emphasises conceptual understanding, the other part leverages skills and practice. This bears an interesting resemblance to the NRC finding regarding the interplay of phonics and whole-language approaches to children's development of reading comprehension (Snow, Burns and Griggin, 1998).

Secondly, teacher development (*kounaikenshuu*) tends to be idiosyncratic in Japanese secondary schools. Instruction in the later secondary school years is not centrally linked to the process of lesson study. This may be due, in part, to departmental specialisation in Japanese secondary schools somewhat parallel to the Irish setting. It is also probably due to the pressure imposed by university entrance examinations and the focus on examination preparation (Yoshida, 1999). This is again quite similar to the Irish setting.

Thirdly, some view lesson study as a way of decreasing primary school teacher autonomy (anonymous presenter at a meeting funded by the National Science Foundation, June 2003). This rendering of the use of lesson study is in stark contrast to the manner in which lesson study is presented by most, if not all, authors. In sumarry lesson study is culturally bound and needs to be understood in the context of the Japanese culture. Some might argue that before we can fully understand and use the lessons of lesson study we need to know what fundamental social values are reflected in the different education systems of Ireland and Japan. Further, we should also know what are the intrinsic and extrinsic incentives motivating Irish and Japanese students.

Additionally, it would be wise to acknowledge the different definitions of democracy as applied to education in Ireland and Japan. In the Republic of Ireland, recognition of different talents is consonant with democracy. In Japan, 'equal access' based on standardised scores on entrance examinations is the implied, culturally-held definition of democracy. So what does this mean for teaching and learning in Ireland? Nevertheless, we think there are some important continuing professional development lessons to be learned from lesson study. In particular, lesson study highlights the importance of teachers' deep knowledge of content and pedagogical content knowledge (knowledge of how to represent subject matter in support of student learning). Furthermore, lesson study is a vivid and powerful image of how teachers can create contexts for collegial discussions of pedagogical practices.

Finding answers to these questions is quite daunting and takes us away from a central tenet of this document. That is, we believe that Irish teachers can learn a lot from the engaging in the process of lesson study. When used appropriately it is a very strong professionalising activity that improves both teaching and learning. However, we do not believe that lesson study alone will move Irish education forward, nor should it.

In embracing lesson study we also believe that the iterative process can be shortened if the research literature of mathematics education paid more attention to issues of practice, and teachers in turn attended to this 'new' literature (see, for example, recent work in variation theory approaches to the learning of mathematics, or the work of Lesh and others on model-eliciting mathematics problems). Embracing lesson study provides a unique opportunity for improvements in mathematics instruction and in the profession of teaching in Ireland, while also building bridges between research and practice in Irish schools.

2.5 Conclusion

In this chapter we have reviewed current trends in understanding and enhancing mathematics teaching. The video and lesson study examples provide new and important ways of engaging teachers in the nature of their professional practice, as well as providing important insights for policy makers and researchers. We highlighted a number of key findings in relation to international trends in mathematics education:

- The increasingly widespread use of various types of video studies to understand mathematics teaching.

- The power of video in promoting understanding of the nature of mathematics teaching, with a specific focus on a key component of teaching, that is, the lesson which is a daily reality in all teachers' lives.

- The move away from elevating Japanese teaching of mathematics as the most desirable way of improving mathematics (TIMSS 1995 video study) to an understanding of how there are many different ways of promoting high-quality mathematics (TIMSS 1999 video study). Nevertheless, there are important lessons to be learned both from the video study of Japanese mathematics teaching and Japanese lesson study as a model of subject-specific professional development.

- The role of new technologies in providing video-generated archives of teaching that can be used for future research and/or teacher professional development.

- Lesson study as a powerful model of teacher professional development rooted in Japanese culture.

- The limitations of borrowing teaching and/or professional development initiatives from another culture without a thorough understanding of both their original contexts and the constraints and affordances of introducing these in another culture.

We finish the chapter with one important insight from the TIMSS 1999 video study. Education Week, a free US-based online magazine for educators, reporting the launch of the report of the TIMSS 1999 video study, drew attention to differences in teaching emphases between the relatively low-scoring USA and high-scoring countries.

> *The study found that American middle school teachers use teaching approaches similar to those of their counterparts in higher-achieving countries. But the U.S. teachers, the report says, omit one critical ingredient: the underlying mathematical ideas that help students understand how the skills they're learning are part of a logical and coherent intellectual discipline. "Higher-achieving countries focus on developing conceptual underpinnings of the problems," said James Hiebert, a professor of education at the University of Delaware, in Newark, and one of the researchers. (Education Week, 2 April 2003, www.edweek.org)*

The key insight from Hiebert's observation is that the nature of the links between procedural skills and conceptual knowledge in classroom practice is a critically important dimension of high-quality mathematics education. The relationship between these two important dimensions of teaching has been the focus of considerable research in the learning sciences. In the next chapter, we look at the various perspectives on learning, each with a somewhat different emphasis in its approach to the relationship between procedural skills and conceptual knowledge.

Chapter 3

Cultures of learning in mathematics education: rethinking teaching and assessment

3.1 Introduction

Some of what the schools have adopted from the research disciplines has impeded deep learning and widespread achievement. The belief system in schools is consistent with beliefs held in the larger culture. For example, only recently have people come to believe that there might be alternative ways to think about the conditions of learning apart from individual capabilities and differences. Research concerned with individual differences has been held captive by its own ideas and the ideas of the larger culture. Breaking out of the box to imagine new possibilities for thinking and learning is both difficult and necessary.

Greeno and Goldman,
Thinking Practices in Mathematics and Science Education, 1998

Mathematics educators are moving from a view of mathematics as a fixed and unchanging collection of facts and skills to an emphasis on the importance in mathematics learning of conjecturing, communicating, problem solving and logical reasoning.

Lester, Lambdin and Preston,
Alternative Assessment in the Mathematics Classroom, 1997

How do different theories of cognition and learning help us think about mathematics education? How have theories of learning shaped assessment in mathematics education? Is there a consensus on how people learn? Can learners achieve mastery of routine procedures without actually understanding mathematical tasks? Under what set of teaching conditions are information and communication technologies likely to enhance mathematics learning? Why do learners typically suspend sense making in interpreting word story problems in school? Why do students typically give up trying to solve a mathematics problem if they cannot solve it in five minutes? What is the best way to teach basic skills and higher order concepts –

skills first then concepts, or vice versa? What is the role of learners' intuitive sense of important school concepts in the promotion of in-school learning? What is the role, if any, of learners' out-of-school experience in the promotion of school mathematics? What makes a mathematics activity realistic for students? These are among the questions that have been at the heart of research into mathematics education.

In addressing these questions we focus on different learning frameworks and philosophies that underpin mathematics education. In particular, we focus on the Realistic Mathematics Education (RME) movement and situated cognition, both of which underpin PISA's mathematical literacy framework (OECD, 2003; Romberg, 2000). Over the last hundred years, the behaviourist, cognitive and socio-cultural approaches to learning have been influential. Of these, the behaviourist and cognitive have been the most researched, with socio-cultural now becoming a widely researched and influential approach. However, people have been trying to understand learning and the mind for centuries and many of the assumptions underlying each of these three perspectives have a long history in philosophy (Case 1996; Greeno, Collins and Resnick, 1996). For example, key ideas underpinning the behavioural approach can be traced back to ideas of John Locke and his associationist perspective on the human mind. Cognitive psychology, particularly Piagetian constructivism, draws on Kant's conceptions of a priori mental categories. The various theories under the socio-cultural umbrella (e.g. situated cognition) have roots in Marx and Hegel's socio-historical epistemologies that locate learning in the social and material history of cultures into which learners become enculturated to a greater or lesser degree. Despite the long debate on cognition and learning, there is no agreement about the exact workings of the mind, in the sense that one theory dominates the discourse. However, a certain consensus has emerged in the last two decades that social and cultural influences on cognition and learning have been neglected (for a discussion see Bruner, 1996). This has resulted in the realisation by educators and researchers of the overemphasis on the isolated learner and his/her capability rather than considering the learner embedded within formative and potentially supportive social and cultural settings in which ability can be developed in various ways (Conway, 2002). One could make a strong case that the emphasis on individual capabilities has been a particularly strong feature of mathematics education. As Greeno and Goldman (1998) have argued, the focus on learners' individual capabilities in mathematics and science education has resulted in an under-utilisation of learners' out-of-school knowledge, unnecessary lowering of expectations about what learning is possible in classrooms, underestimation of the role of peers in contributing to learning, and a reliance on teaching strategies which overly compartmentalise teaching/learning activities.

Assumptions about learning mathematics are deeply embedded in the culture of schooling and evident in textbooks (see chapter one), parents' and educators' everyday or folk theories of learning (Olson and Bruner, 1996), modes of assessment, and the daily rhythm of lesson planning and lesson enactment. Olson and Bruner (1998) argue that, 'the introduction of any innovation will necessarily involve changing the folk psychological and folk pedagogical theories of teachers - and to a surprising extent, of pupils as well' (p. 11). The recognition of the necessity of paying more attention to both our folk/everyday and academic theories of learning is evident in the high profile being accorded research on learning over the last decade, as agencies (e.g. OECD, APEC, UNESCO) and governments consider optimal ways to enhance every citizen's learning in a period of rapid social, cultural, economic and educational change.

The first section of this chapter provides an overview of approaches to learning that have influenced mathematics education over the last hundred years, particularly behaviourist, cognitive and socio-cultural approaches, focusing on how each addresses procedural skills and conceptual knowledge, the development of learners' ownership of learning, and assessment. We then discuss 'realistic mathematics education' (RME) whose origins are in mathematics education rather than the learning sciences. However, in discussing the appeal of RME we note its similarities with the 'social turn' in learning theory. The next section of the chapter reviews recent interest internationally in brain-based research and its potential to inform developments in mathematics education. The final section of the chapter addresses the increasingly important role accorded learning to learn in international discourse on educational goals. We discuss learning to learn, drawing on cognitive and socio-cultural research on self-regulated learning.

3.2 Different approaches to learning in mathematics education

This section of this chapter provides an overview of three approaches to learning and also focuses on the very significant contribution of the Realistic Mathematics Education (RME) movement (see Section 3.2) to contemporary debates on mathematics education. As we have noted earlier (see Chapter 1), RME has become very influential in mathematics education, given that and situated cognition underpin the PISA mathematical literacy framework. The adoption of RME and situated cognition represents what Romberg (2000) calls a distinct 'epistemological shift' in mathematics education. At the heart of this epistemological shift are (i) a reconceptualisation of the relationship between procedural skills and conceptual knowledge and how they are understood to work together in 'applying' maths, and (ii) a recognition of the neglected role of the social and cultural setting as both the source of learning and the arena within which it is applied.

Historically, according to De Corte *et al*. (1996), approaches to teaching and learning in mathematics have emanated from two sources: (i) mathematicians and/or mathematics educators like Freudenthal, Polya, Poincaré, etc., and (ii) research in the learning sciences – primarily, cognitive/educational/developmental psychology and cognitive anthropology, and more recently cognitive neuroscience. Before we outline the three perspectives, we address the question as to whether or not academic or folk/everyday theories of learning matter in relation to understanding mathematics education practices.

At one level, theories of learning appear far removed from classroom practice in the sense that it may not be at all clear about the particular approach to learning underpinning specific practices, as these practices may have been shaped by a variety of influences such as curricular cultures, textbooks and examination traditions. However, a more detailed analysis often reveals some important links between teachers' beliefs about learning, their practices and student learning. Morine-Dershimer and Corrigan (1997), for example, interviewed student teachers in California regarding how they thought about mathematics learning and teaching. Based both on the interviews and videos of the student teachers' classroom teaching practices, they identified two groups: the first group had beliefs focused on getting knowledge and ideas across to students in an efficient and timely fashion; the second group emphasised the importance of attending to students' prior knowledge and experience and how they might integrate this into their lessons. Perhaps what was most interesting was how these beliefs were evident in quite different sets of teaching practices by the two groups of students; that is, they taught in ways congruent with their stated beliefs about mathematics teaching and learning.

A more recent longitudinal German study, undertaken by researchers from the Max Planck Institute, provides further evidence that teachers' beliefs about learning make a difference to student achievement. Staub and Stern's 2003 study involved 496 students in 27 self-contained 2nd and 3rd grade class in Germany in order to assess the degree to which teachers' beliefs were consistent with behaviourist/direct transmission or cognitive-constructivist views of teaching and learning. They also observed classroom teaching using an observation protocol to ensure the consistency and reliability of observations, and tested students on both routine procedural arithmetic and word story problem-solving measures. As Staub and Stern's study adopted a longitudinal research design, utilised robust statistical analyses (i.e. multi-level modelling which allowed the researchers to assess the nested effects of teachers on students and student growth over time), and focused on student learning with links back to teachers' beliefs and practices, we note the main findings (Raudenbush and Bryk, 2002). First, as in the Dershimer and Corrigan study, there were clear links between teachers' beliefs and their practices. Second, teachers who held the constructivist orientation provided more frequent conceptually-oriented learning opportunities for students that resulted in higher scores on both procedural and problem-solving tasks than teachers working from a behavioural orientation. This result is important, as Staub and Stern note, in that 'contrary to our expectations, there was no negative impact of a cognitive constructivist orientation on arithmetic tasks' (p. 353). Staub and Stern's study supports similar findings from an earlier US-based study by Peterson, Carpenter and Fennema (1989). Discussing the implications of their study, Staub and Stern note that children's prior mathematical achievement and ability have a stronger impact on their achievement than teachers' beliefs; nevertheless they argue that the cumulative impact of teachers' learning-related beliefs/practices over many years of schooling represent a significant impact on student learning.

Three perspectives on learning and assessment in mathematics education

Behavioural, cognitive and socio-cultural theories are the three schools or perspectives on learning that have had a significant impact on mathematics education over the last hundred years (De Corte, *et al.*, 1996; Greeno, Collins and Resnick, 1996). In this section, we address how each has affected approaches to mathematics education with particular reference to how each addresses

- procedural skills and conceptual knowledge
- their relative emphasis on the social and cultural dimensions of learning
- developing learners' ownership and responsibility for learning (with particular focus on self-regulated learning)
- assessment.

At the outset, we want to point out that of these three perspectives, two of them, the behavioural and cognitive, have had a more marked impact to date than the sociocultural perspective. However, in the last two decades a very significant shift towards socio-cultural theories reflects a wider movement in social science towards a more social and cultural understanding of human conduct. This is evidenced, as we noted earlier, in the adoption of situated cognition by the OECD as its preferred perspective on learning.

Definitions of learning

While a detailed exposition of the differences between these three perspectives on learning is beyond the scope of this report, we note how each understands learning and assessment in the context of mathematics education. In the behaviourist tradition learning is change in behaviour; in the cognitive tradition learning is change in thinking; and in the socio-cultural tradition learning is change in participation. These widely diverging definitions of learning draw the attention of mathematics teachers and curriculum designers to different sets of questions regarding assessment.

Learning theories and assessment practices

The three perspectives have different views of assessment. In the behaviourist tradition assessment involves an appraisal of how well learners have mastered component parts of skills, progressing from low-level to high-level skills. In the cognitive tradition, because learning is defined as change in thinking, the focus of assessment is on understanding change in learners' conceptual understanding, learning strategies and learners' thinking over extended periods, often involving real-world contexts. In the socio-cultural tradition, because learning is defined as engaged participation with agency, assessment focuses on learners' engagement with real-world tasks/problems, typically over an extended period of time. The cognitive and socio-cultural perspectives overlap in their focus on the value of real-world contexts and both involve an interactive component in assessment. In mathematics examination and assessment systems, the combined effect of the behavioural focus on decontextualised skills and the 'new' mathematics focus on abstraction have led to examinations and assessments that often pit the isolated learner against quite abstract tasks bearing little relation to real-world challenges of a mathematical nature.

Behaviourism: direct teaching followed by controlled practice

> *Conceptions of learning as incremental (with errors to be avoided or immediately stamped out), of assessment as appropriately implemented by reference to atomistic behavioural objectives, of teaching as the reinforcement of behaviour, of motivation as directly mediated by rewards and punishments, and of mathematics as precise, unambiguous, and yielding uniquely correct answers through the application of specific procedures remain prevalent in folk psychology and, as such, represent the legacy of behaviourism. Moreover, the development of behaviourism led to persisting views of learning hierarchies within mathematics, with harmful effects, according to L. B. Resnick. (1987a, pp. 48-49) (De Corte et al., 1996, p. 493)*

Learning and teaching in the behaviourist tradition

De Corte *et al.*'s summary of the impact of behaviourist thinking on mathematics education describes how it shaped conceptions of teaching, learning and assessment, and lent support to folk theories of teaching, learning and assessment. As we noted earlier (see chapter one), the persistence of and over-emphasis on learning hierarchies has been a distinct feature of mathematics education, and this is especially so for students deemed less able (Means and Knapp, 1991; Conway, 2002). Behaviourism put a premium on three basic pedagogical strategies: breaking down tasks into small and manageable pieces, teaching the basics first, and incrementally reinforcing or rewarding observable progress. From this perspective, knowledge can be seen as a hierarchical assembly or collection of associations or behavioural units.

Perhaps the most widely recognised and intuitively appealing implications of the behavioural perspective are its recommendations for designing teaching. These are the simplification and sequencing of tasks into discrete hierarchical steps and reinforcing successful approximations of desired activity. In sumarry, the hallmarks of behaviourism are presenting learning in small steps, in the simplest possible form, sequencing tasks in a hierarchy from the simple to the complex, and rewarding successful observed behaviours. Two problems associated with this approach to teaching are the assumption of 'vertical transfer' and the decomposition of activities such as reasoning and problem solving, resulting in a lack of task wholeness and authenticity. Vertical transfer assumes that learners will assemble the various associations or connections lower down on the learning hierarchy, and integrate these in order to eventually engage in higher order tasks. This vertical transfer problem is interwoven with what critics view as the lack of task authenticity when teaching is designed from a behavioural perspective. Thus, rather than involving learners in the full authenticity of say mathematical problem solving, a behavioural perspective focuses on teaching the fundamental elements (in the case of maths, basic arithmetic algorithms) prior to the more complex and contextually framed elements such as story problems (Koedinger and Nathan, 2004).

As De Corte *et al.* (1996) argue, behaviourism has had a powerful influence on views of teachers' and curriculum designers' understanding of learning and how best to assess it. In relation to assessment, behaviourism focuses on breaking down content into its constituent parts and assessing each part based on the assumption that, once it is known that learners can demonstrate their skill on the parts, the more integrative higher-order skills will flow naturally from these sub-skills. In contemporary context, the behaviourist perspective is evident in Precision Teaching[8] (Lindsley, 1990; Lindsley, 1991), and much drill and practice or computer-assisted instruction (CAI) software in mathematics (e.g. Mathblaster). In the case of drill-oriented computer software it is, typically, premised on a mastery framework of learning where students are rewarded for correct answers (typically 4 out of 5), and may then proceed to the next level of difficulty in the prescribed learning hierarchy.

Relationship between skills and development of concepts

One of the key features of behaviourism is its emphasis on teaching and assessing the component parts of skills prior to teaching the aspects of skill further up the learning hierarchy. In this sense, from a behavioural perspective learning involves the development of many component skills.

Learners' social and cultural background

Learners' social and cultural background is largely irrelevant from a behavioural perspective since what matters in terms of capacity to learn is the strength of learner's prior stimulus-response pairings related to key learning objectives. In fact this asocial view of the learner was one appealing feature of behaviourism in that it heralded and promised predictable programmed learning as a possibility for all learners.

8 Precision Teaching, as form of Applied Behaviour Analysis (ABA), is based on operant conditioning principles and focuses on pacing learners based on observations of learner behaviour on incrementally structured learning tasks. It has been widely used in special education, mathematics education and has also been employed to design computer-aided instruction (CAI).

Development of self-regulation

The development of self-regulation focuses on self-instruction with attention to identifying reinforcements that will strengthen desired behaviour. For example, Belfiore and Hornyak (1998) describe a homework completion programme involving a routine checklist to assist learners in self-instruction:

- Did I turn in yesterday's homework?
- Did I write all homework assignments in my notebook?
- Is all homework in my homework folder?
- Are all my materials to complete my homework with me?
- etc.

Learners can be helped in identifying and using self-reinforcement when an agreed criterion or standard has been reached by the student, e.g. when a certain number of mathematics problems have been completed.

In summary, a behavioural approach is consistent with a direct or transmission-oriented approach to teaching, and in the context of mathematics could provide support for the direct expository type of teaching associated with the new mathematics (see chapter one) movement where the role of the teacher is to convey/transmit the logical, hierarchically structured nature of mathematics. Behaviourally-inspired teaching focuses on supporting individual learners' movement through a prescribed hierarchy, from simple to more complex learning skills. The exact way in which learners vertically transfer and integrate basic skills with more complex in order to solve problems in mathematics is an aspect of learning that is not addressed very well in the behavioural approach.

Cognitive: promoting active learning and problem solving

> *From a contemporary perspective, Piaget's main contribution to mathematics education was his demonstration of the complexity of children's thinking and the qualitative differences in thinking at various stages of development. He is acknowledged as a major inspiration of the radical shift to the conception of the child as an active constructor of knowledge.*
>
> **(De Corte *et al.*, 1996)**

Learning and teaching

The emergence of cognitive perspectives on teaching and learning in the 1960s marked a very significant break with the traditions of behaviourism (Gardner, 1985; Collins, Greeno and Resnick, 1996). In the context of current developments in mathematics education, cognitive perspectives are informative in four ways: (i) their conception of active learning, and assessment instruments congruent with such a view of learning; (ii) the notion of cognitive challenge and how various degrees of cognitive challenge are embedded in all teaching and assessment situations, (iii) a conception of expert problem-solving or competent problem-solving in mathematics; and (iv) the development of significant literature demonstrating the applicability and efficacy of teaching self-regulated learning using various strategies (e.g. cognitive strategy instruction, goal setting, self-assessment, peer assessment, and formative feedback from teachers) for children of all ages and abilities.

Firstly cognitive perspectives represent a distinct break with the epistemology underpinning behaviourism. Resnick conveys the shift in epistemology well, commenting that, 'Learning occurs not by recording information but by interpreting it' (1989, p. 2). Over time, the notion that learning takes place through the active construction of knowledge by the individual learners has gained considerable attention among teachers and curriculum designers, and is now reflected in mathematics curriculum documents internationally. Knowledge, rather than being 'out there' (the basic assumption from behaviourist stance), is constructed by learners' actions on the world. As such, knowledge is made as learners engage with and experience the world.

The cognitive perspective has provided many important insights with which to plan both classroom teaching and assessment. Among the most important of these are that learning is active, learning is about the construction of meaning, learning is both helped and hindered by our prior knowledge and experience, learning reorganises our minds, the mind develops in stages, and learning is more often than not unsettling. Based on these insights, a diverse range of strategies has been developed for classroom practice, many of which have been evident in various textbooks, teacher handbooks and curricular documents over the last thirty years. Much of the appeal of cognitive theories, in the international context, grew out of the desire to move away from didactic and transmission-oriented teaching. Many advocates of active learning would echo Dewey (1933, p. 201), who in his book *How We Think*, in opposition to the didactic nature of classroom teaching at that time, spoke out against 'the complete domination of instruction by rehearsing second-hand information, by memorizing for the sake of producing correct replies at the proper time'. Anticipating some of the arguments and claims made by cognitive and educational psychologists over the last forty years, Dewey argued for the importance of students' active involvement in the learning process and problem-solving as the context within which to learn information.

Secondly in terms of assessment, cognitive perspectives focus on more extended, problem-focused and authentic tasks consistent with learners' mental models of key concepts in mathematics. Cognitive perspectives on assessment have provided complex models of cognition and the types of inference that can be drawn from performance on assessment items. For example, one of the legacies of the cognitive perspective was to draw educators' attention to levels of cognitive challenge embedded in assessment items as evidenced in the development and widespread use of Bloom's Taxonomy with its six levels of thinking representing increasing cognitive demands on learners. Bloom's original *Taxonomy of the Cognitive Domain* (Bloom *et al.*, 1956) is possibly one of the most widely used frameworks in education around the world. As Anderson (2002) commented, '…numerous classroom teachers can still recite them: knowledge, comprehension, application, analysis, synthesis and evaluation. It is not unusual to walk into a classroom and see a poster of the taxonomic categories (hand drawn or commercially produced) on a wall or bulletin board.' Furthermore, the original *Taxonomy* has been translated into 22 languages since its first publication in 1956 (Anderson, 2002). The revised *Taxonomy* (Anderson *et al.*, 2001), like the original, has six levels in the cognitive domain: remember, understand, apply, analyse, evaluate and create. However, the cognitive processes in the revised *Taxonomy* underwent some notable changes: three processes were renamed, the order of two was changed, and the verb form of processes was used instead of the noun form (see *Classroom Assessment: Enhancing the Quality of Teacher Decision-Making*, Anderson, 2003 for discussion of the Revised Bloom's Taxonomy in the context of teacher decision-making and its potential to change curriculum planning and assessment; Conway, 2005d). Consistent with the general principle of assessing cognitive challenge or cognitive complexity, this is a central design feature in the PISA mathematics literacy items with three levels of challenge, or 'competency clusters', to use the PISA terminology. These are: (i) *reproduction*, that is, performing calculations, solving equations, reproducing memorised facts or 'solving' well-rehearsed routine problems; (ii) *connections*, that is, integrating information, making connections within and across mathematical domains, or solving problems using familiar procedures in contexts; and (iii) *reflection*, that is, recognising and extracting the mathematics in problem situations, and using that mathematics to solve problems, analysing and developing models and strategies or making mathematical arguments and generalisations (OECD, 2003, p. 49; Close and Oldham, 2005, p. 176; Cosgrove *et al.*, 2005, pp. 7-9).

Thirdly cognitive perspectives provide a model of competent problem-solving and strategies for teaching it in mathematics and other domains. As De Corte *et al.* (1996) note:

> *A major finding of the analysis of expertise is that expert problem solvers master a large, well organized, and flexibly accessible domain-specific knowledge base… Indeed, it has been shown that experts differ from novices in that their knowledge base is better and more dynamically structured, and as a consequence more flexibly accessible. (Chi* et al.*, 1988, p. 504)*

One of the most fruitful and practical ideas emerging from a number of strands in cognitive research is the efficacy of teachers modelling and making explicit the strategies they adopt in understanding and solving problems via teacher 'think alouds' and/or providing 'think sheets' for students to assist them in monitoring and controlling their own thinking while solving problems. Considerable research suggests that teachers rarely use think alouds and other strategies that model and make explicit complex and expert problem-solving. This is especially troubling as such strategies have been demonstrated to be effective with lower-achieving students in both primary and post-primary settings.

Relationship between components: lower-order and higher-order thinking

While it might be overstating it to say that the jury is out within the cognitive perspective as to the relative merits of adopting a bottom-up or top-down stance on the relationship between lower-order and higher-order thinking, this issue has been contested in the cognitive tradition. As Greeno, Pearson and Schoenfeld (1997) note:

> *According to some analyses, the elementary aspects of achievement - routine skills, facts and concepts - are prerequisite to learning more complex 'higher order' aspects of achievement. However, this is a disputed view. An alternative is that strategic, meta-cognitive, and epistemic aspects of achievement are more fundamental than detailed procedures and routine to effective intellectual functioning. (p. 159)*

For example, if students have not had experiences where they have learned to think mathematically (a higher-order strategic resource), it is unlikely that they will deploy computational skills in a timely and appropriate fashion. While it has been agreed that both lower and higher-order skills are important, it is vital, from a teaching perspective, to have some clarity on where the emphasis should lie: teaching basic skills first followed by higher-order skills, or embedding basic skills within complex problem-oriented situations where learners grapple with multiple levels simultaneously.

Development of self-regulation

The development of strategic, purposeful self-regulated learning (SRL) is one of the hallmarks of the cognitive perspective. SRL has become an increasingly important policy focus in mathematics education internationally. Its focus is on learning to learn strategies through a number of lines of research, including cognitive strategy instruction, motivation (particularly goal theory and related goal-setting strategies), and formative feedback (teacher, peer and self-assessment, and self-regulation). Again, these ideas from learning sciences have filtered into policy-making and international assessments. This is clear from the focus on self-regulation as a core competency in APEC policy documents (APEC, 2004) and the inclusion of items in PISA on self-regulation, as well as the development within PISA of a theoretical framework for understanding self-regulated learning as a critically important cross-curricular competence (Baumert, *et al.* 1999).

While behaviourist and cognitive theories are based on very different assumptions about learning, knowing and intelligence, and have very different implications for classroom practice, they share one defining feature, namely their focus on the individual learner with little emphasis on the cultural and historical context of mathematics learning. As Conway (2002) notes:

> *Rather than viewing the learner as part of family, community and social group embedded in a particular time and place, both the behavioural and cognitive perspectives portray learning as primarily a solo undertaking. Thus, what is neglected, in this focus on the solo learner, is how the learner is situated amidst levels of guidance by more knowledgeable others, nurtured via social support, influenced by peer norms, and shapes and is shaped through engaging in communication with other humans and various media within evolving cultural and historical circumstances. (p. 76)*

In a similar vein, De Corte *et al.* (1996) claim that, 'while arguing that ideas about social construction are integral to Piaget's genetic epistemology, it is acknowledged that there are grounds for the caricature of a Piagetian, as Youniss and Damon (1992) note, as the "apocryphal child who discovers formal properties of things, such as number, while playing alone with pebbles on the beach"' (p. 268). These and other criticisms of cognitive perspectives on learning and mathematics led researchers to look toward more culturally embedded conceptions of learning. Socio-cultural perspectives offered such a view and are best seen as a cluster of related theories, of which situated cognition is one of a range of possible theories.

Socio-cultural perspectives: engaged participation

Socio-cultural theories are consistent with constructivist learning frameworks although they adopt a more socially embedded view of learners than Piagetian/individual constructivist models (Prawat and Floden, 1994) and are most closely associated with the writings of Dewey and Vygotsky in the early part of the twentieth century. More recently, various authors writing from a situated cognition perspective (Brown, Collins and Duguid, 1989) have drawn upon the cultural and social views of learning in the writings of Vygotsky (his Russian contemporaries Luria and Leontiev) and Dewey. Socio-cultural theories assert that the mind originates dialectically through the social and material history of a culture which a person inhabits (Vygotsky, 1978). The emergence of socio-cultural perspectives in mathematics education reflects a wider 'social' turn in understanding learning in education (Lerman, 2002). This position is in marked contrast to the view that the mind has its primary origin in the structures of the objective world (behaviourist position) or that it has its origin in the order-imposing structures of the mind (cognitive perspective) (Case, 1996).

Learning and teaching

What are the implications of these assumptions for teaching and assessment in mathematics? In terms of pedagogy, rather than focusing on individualized teaching, socio-cultural theories put a heavy emphasis on fostering communities of learners (Prawat, 1992), which provide not only opportunities for cognitive development but also the development of students' identities as numerate members of knowledge-building communities. Brown (1994) outlined a coherent set of principles underpinning the notion of a 'community of learners' as well key strategies for its implementation. These principles are

- academic learning as active, strategic, self-motivated and purposeful
- classrooms as settings for multiple zones of proximal development through structured support via teacher, peer and technology-aided assistance of learners
- legitimisation of individual differences
- developing communities of discourse and practice, and
- teaching deep conceptual content that is sensitive to the developmental nature of students' knowledge in particular subject areas.

The integrated implementation of these five principles forms the support for the emergence of communities of learners in classroom settings (Brown, 1997). Two of the key constructs in Fostering a Community of Learners (FCL) are classroom discourse patterns and participation structures.

Firstly, based upon the insight that much academic learning is active, strategic, self-motivated and purposeful, Brown emphasised how FCLs ought to focus on the development of students' capacity to think about thinking, that is to engage in metacognition. As such, a key feature of FCLs is the promotion of a culture of meta-cognition, directed towards the development of learning to learn strategies. In the case of mathematics, for example, teaching students self-monitoring strategies becomes an essential part of teaching (see Schoenfeld, 1985). Verschaffel, De Corte, Lasure *et al*. (1999) describe this in terms of modelling various heuristic strategies for students in the course of problem-solving.

- Build a mental representation of the problem.
  Draw a picture; make a list, scheme or table; distinguish relevant from irrelevant; use your real-world knowledge.

- Decide how to solve the problem.
  Make a flow-chart; guess and check; look for a pattern; simplify the numbers.

- Undertake the necessary calculations.

- Interpret the outcome and formulate an answer.

- Evaluate the solution in terms of the problem.

Secondly, drawing upon Vygotsky's Zone of Proximal Development (ZPD), that is, the difference between what a learner can do by themselves versus what they can do with the assistance of another person and/or tool, Brown emphasised teaching toward the upper rather than lower bounds of students' competence. Much contemporary pedagogical practice, she claims, focuses on matching teaching with students' existing levels of competence – that is, the lower bounds of competence. In many mathematics classrooms, with their focus on students' individual capabilities, there are few opportunities for students to learn from each other (see TIMSS video study findings in chapter two). Japanese 8th grade mathematics classrooms were a notable exception in that teachers often started class by posing a complex written problem on the blackboard, telling students to think and consult with one another, after which students shared potential solutions with the whole class. In marked contrast, USA and German mathematics teaching focused on providing explicit fast-paced, teacher-led instruction in order to prepare students for individual practice with algorithms on a series of almost identical mathematical problems (Stigler and Hiebert, 1999).

Thirdly, FCLs are intended to value and nurture students' diverse cultural perspectives, support multiple entry points into subject matter (via art, music, drama, technology, story, text, etc.), and foster diversity in the distribution of expert knowledge. The emphasis on entry points in Brown's FCL shares similarities with Gardner's emphasis on entry points (1999) and RME's focus on rich contexts for engaging students in horizontal mathematising. For example, Freudenthal (1991) stressed the importance of rooting mathematising in rich contexts such as location, story, projects, themes, and newspaper clippings (pp. 74-75).

Fourthly, FCLs are premised on the belief that higher-level thinking is an internalised dialogue. Based on this premise, classrooms ought to be discursive environments where students can engage in conjecture, hypothesis testing, and speculation centred around 'hot' mathematical ideas in much the way mathematicians engage with mathematics.

Fifthly, in FCL one of the challenges for teachers is to teach deep conceptual content that is sensitive to the developmental nature of students' mathematical knowledge. One of the major findings from cognitive research over the last twenty years is that children and adolescents 'have a strong intuitive understanding of concepts that can support instruction, which in turn can further advance their conceptual understanding' (Greeno, Pearson, and Schoenfeld, 1997; see also Gopnik, Meltzoff and Kuhl, 1999 for a very readable account of emergent conceptual understanding in children, and Mix, Levine, Huttenlocher, 2002 for an account of the emergence of quantitative reasoning in infancy and early childhood). This is in marked contrast to the over-generalisation of the Piagetian perspective, which often resulted in teachers underestimating students' thinking capabilities. As De Corte *et al.* state, 'we doubt that educational practice needs to be guided very strongly by ideas about the development of general schemata of logico-deductive operations in children's reasoning' (1996, p. 18).

Relationship between basic skills and higher-order skills: lessons from cognition in the wild

Whereas there is disagreement in the cognitive perspective as to whether bottom-up or top-down teaching strategies are optimal, the socio-cultural perspective puts a strong emphasis on teaching/learning basic skills within the context of authentic, real-world teaching situations. Among the sources of evidence cited in support of this strategy, which is completely at odds with the behavioural bottom-up, skills-first stance, are a number of studies of how people of all ages learn complex skills in out-of-school contexts. These have investigated the way in which children and adults learn complex quantitative reasoning and problem-solving when tasks are authentic and provide opportunities for different levels of support and slowly increasing levels of participation (Lave, 1988). For example, research by Nuñes, Schleiman and Carraher

(1993) showed how children in Brazil who were working as street vendors had developed complex algorithms to calculate price and quantity in a fast and accurate fashion. The same children when presented with similar tasks in symbolic form in school situations performed well below their on-street level: their 'school scores' were between a quarter and half of their 'street scores'. The researchers in this study drew a number of conclusions including the following: school did not utilise these children's out-of-school mathematical knowledge; the children had developed sophisticated algorithms, often supported by the active use of materials in specific patterns to develop fast and accurate calculations; and the children learned basic computation within more complex problem-solving activities.

Development of self-regulated learning

The cognitive perspective provides valuable evidence of the efficacy of strategic and purposeful learning, and that such strategic and purposeful engagement can be taught to children and adolescents of varying abilities. The socio-cultural perspective builds on these insights by emphasising the activity settings within which such self-regulatory skills can be developed (Engeström, 1999; Allal and Saada-Robert, 1992; Allal, 2005). Rooted in a socio-cultural perspective, Allal has (Allal, 2005) developed a multi-level regulation framework in which she argues that regulation of learning must be addressed at school, classroom and individual student levels. Thus, in order for students to learn effective self-regulation skills there must be congruence between all three levels.

Three views of assessment

According to their different definitions of learning, each of the three approaches projects a different message about what is important in assessment. From a behavioural perspective, the assessment is concerned with facts, procedures, routines and skills. They are paramount because they are viewed as prerequisites for more advanced skills. Consequently, the assessment emphasis is on components of skills prior to composite skills. Consistent with the focus on learning hierarchies from simple to more advanced skills, assessments are often characterised by hierarchies where the test items in general, or within multiple items on sub-sections of a test, progress from simple to more complex challenges. Thus the behavioural emphasis on skill decomposition is reflected in both teaching and assessment.

A cognitive perspective on assessment emphasises the appraisal of a wide range of knowledge, including conceptual understanding, strategies, meta-cognition (i.e. thinking about thinking), and beliefs. Given the wide-ranging nature of the target of assessment in the cognitive perspective, a similarly wide range of assessment formats is necessary to reflect the different cognitive processes. For example, conceptual understanding and mathematical problem-solving are difficult to assess in test items that only ask the student to produce an answer that can only be marked as correct or incorrect. Assessment of the more challenging thought processes typical of the cognitive perspective can often only be appraised through open-ended response items, extended response items and/or portfolios where students can demonstrate their thinking processes over extended periods of time. Greeno, Pearson and Schoenfeld (1997) characterise the shift in emphasis from behavioural to cognitive perspectives in terms of a move from 'focusing on how much knowledge someone has to providing adequate characterization of just what is the knowledge someone has' (p. 13). A description of the nature of expert learners' knowledge as integrated, flexible and dynamic (Bransford, Brown and Cocking, 2000) has been one of the most valuable contributions of the cognitive perspective to understanding human competence. Consequently, the assessment of expertise ought to provide opportunities to appraise such knowledge. From a cognitive perspective, test/examination items that merely require students, in mathematics or any other subject, to deploy well-rehearsed procedures in solving predictable problem types do little to assess the degree of learners' expert knowledge. Open-ended and extended response type assessments allow assessors to see the variety of ways in which students solve problems (e.g. some may do so more intuitively than others, who may rely on careful step-by-step analysis). The key point here is that expertise is characterised by flexible and generative thinking which will often remain invisible unless assessments are suitably authentic and open-ended to bring forth such thinking (Greeno, Collins and Resnick, 1996; Bransford, Brown and Cocking, 2000).

The socio-cultural perspective on assessment is radically different in its focus from almost all contemporary school assessments. Assessment from a socio-cultural standpoint focuses on learners' capacity to participate in particular activities in order to demonstrate competence in disciplinary ways of knowing. In the case of mathematics, the capacity to engage in problem posing, formulate mathematical models, test such models in symbolic form, and when necessary reflect on the meaning of solutions for the real world, characterises the work of mathematicians. Socio-cultural assessments seek to assess participation as it is embedded in meaningful activities since it is assumed that learners' knowledge is intimately linked with various tools and social supports within such activities. The socio-cultural preference for real-world performance assessment seems idealistic and impractical when held up against the dominant assessment formats used in almost all large-scale mathematics assessments. A sociocultural perspective does not attempt to assess sub-skills in isolation (as in the behavioural model), but assumes that performing complex skills which require the meaningful deployment of relevant routines and procedures, as resources, is a far better way of assessing learners' actual integration of such procedural and factual resources. Assessments of the type preferred from a socio-cultural perspective are rarely seen in mathematics. However, not utilising such assessments seems very much like trying to assess a good band by sitting them down to perform an on-demand paper-and-pencil test rather than asking them to play some music! Typically, large-scale mathematics testing/examinations rarely if ever ask students to perform the mathematical equivalent of playing in a band or even as soloists.

Changing views of assessment: a three-part learning-based model

In this section, we build on some of the aforementioned key developments in the learning/cognitive sciences and also include concurrent developments in the field of measurement, which together are beginning to change the range and feasibility of assessment options for schools and educational policy makers. One feature of educational change we have noted earlier is the persistence of old forms of assessment despite ambitious new definitions of learning (Broadfoot, 2001; Bransford, Brown, and Cocking, 2001). This lag remains a real challenge in improving teaching in key curricular areas.

Educational assessment seeks to determine how well students are learning and is an integral part of the quest for improved education. When used appropriately, it provides feedback to students, educators, parents, policy makers and the public about the effectiveness of educational services. With the movement over the past two decades towards setting challenging academic standards and measuring students' progress in meeting those standards, especially in mathematics (see NCTM, 2000), educational assessment is playing a greater role in decision-making than ever before. In turn, education stakeholders are questioning whether current large-scale assessment/testing practices are yielding the most useful kinds of information for informing and improving education. For example, classroom assessments, which have the largest potential to enhance instruction and learning, are not being used to their fullest potential (Black and Wiliam, 1998a; Black and Wiliam, 1998b).

The US National Research Council (NRC) committee argues cogently that advances in the learning and measurement sciences make this an opportune time to rethink the fundamental scientific principles and philosophical assumptions serving as the foundations for current approaches to assessment. Advances in the learning/cognitive sciences have broadened the conception of those aspects of learning that it is most important to assess, and advances in measurement have expanded the capability to interpret more complex forms of evidence derived from student performance.

In the report *Knowing What Students Know* (Pellegrino, *et al*, 2001) the committee explains that every assessment, regardless of its purpose, rests on three pillars: a model of how students represent knowledge and develop competence in the subject domain, tasks or situations that allow one to observe students' performance, and an interpretative method for drawing inferences from the performance evidence obtained through student interactions with the chosen domain tasks. These three elements comprise what the committee refers to as the 'assessment triangle' and underlie all assessments.

All three must be co-ordinated; that is, the three constituent parts of all assessments include ideas about cognition, about observation and about interpretation. Moreover, and most importantly, these three elements must be explicitly connected and designed as a co-ordinated whole. It is this connectedness that distinguishes this model from most offered in mathematics education.

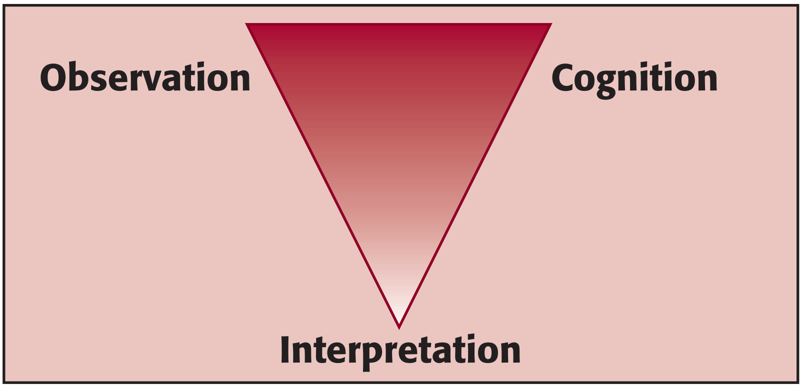


Figure 1: A model for the assessment of learning in mathematics

Traditionally, the mathematical task has been to the fore (see, for example, Kulm, 1990; Lesh and Lamon, 1992; and Mathematical Sciences Education Board, 1993), and discussion of cognition and interpretation has been most often ignored.

- Cognition
  Model of how students represent knowledge and develop competence in the domain.

- Observation
  Tasks or situations that allow one to observe students' performance.

- Interpretation
  Method for making sense of the data.

The committee further asserts that current assessment strategies are, generally speaking, based on highly restrictive models of learning. They note that the model of learning should serve as a unifying construct across the three elements. The model of learning should be based on current knowledge about human cognition, and should serve as the nucleus, or glue, that coheres curriculum, instruction, and assessment (see Gardner, 2006 for a detailed discussion of the increasingly central role accorded learning in framing assessment based on over fifteen years research undertaken by the A** in the England, Scotland Wales and Northern Ireland). Put simply, educational assessment must be aligned with curriculum and instruction if it is to support the learning of mathematics (or any other content). It is striking to again see the parallels here with Cognitively Guided Instruction and lesson study for example. The insights from the Realistic Mathematics Education movement can be seen as providing a very rich understanding of cognition, as well as opportunities for careful observation of students engaging with well-structured tasks - that is, ones likely to provide optimal inferences of what students know and understand.

Despite the plethora of evidence making a case for new approaches to assessment, and a considerable number of small-scale, well-researched projects presenting generally positive findings on the use of alternative assessments more closely linked to contemporary cognitive and socio-cultural conceptions of learning and mathematics education, the adoption of these assessment modes for large-scale assessment in mathematics has been slow, and so far the 'preponderance of assessment, in practice, remains unreformed' (Verschaffel, Greer and De Corte, 2000, p. 116). We provide three examples of initiatives in assessment which put learning more to the fore: (i) assessment within RME, (ii) a statewide reform of the upper post-primary mathematics assessment system in Victoria, Australia, and (iii) NCTM-inspired reforms in classroom assessment in the US.

Assessment within Realistic Mathematics Education (RME)

A detailed account of assessment in RME is available in Van den Heuvel-Panhuizen (1996). As a basic principle, RME (see section 3.2) seeks to design assessment and learning opportunities that are genuine problems - 'rich, non-mathematical contexts that are open to mathematization' (Van den Heuvel-Panhuizen, 1996, p. 19). Van den Heuvel-Panhuizen (1996) distinguishes RME assessment items from traditional word story problems which are often 'rather unappealing, dressed up problems in which context is merely window dressing for the mathematics put there' (p. 20). Thus RME textbooks include practical application problems rather than artificial word story problems. For example, a typical RME application problem reads as follows:

> *Mr Jansen lives in Utrecht. He must be at Zwolle at 9.00 Tuesday morning. Which train should he take? (Check the train schedule.)*

There are number of characteristics of this problem that mark it as different from the traditional word story problems in most mathematics textbooks. Van den Heuvel-Panhuizen (1996) explained the RME rationale for such assessment items as follows:

> *This problem is nearly unsolvable if one does not place oneself in the context. It is also a problem where the students need not marginalize their own experiences. At the same time, this example shows that true application problems have more than one solution and that, in addition to written information, one can also use drawings, tables, graphs, newspaper clippings and suchlike. Characteristic of this kind of problem is the fact that one cannot learn to do them by distinguishing certain types of*

problems and then applying fixed solution procedures. The object here is for the student to place him or herself in the context and then make certain assumptions (such as how far Mr. Jansen lives from the station and how important it is that he arrives at his destination on time). (p. 20-21)

Verschaffel, Greer and De Corte (2000) note the defining characteristics of RME assessments as follows: extensive use of visual elements; provision of various types of materials (e.g. train timetables); all the information may not be provided; there is a general rather than single answer; a focus on relevant and essential contexts; asking questions to which students might want to know the answer; and using questions that involve computations before formal techniques for those computations have been taught. This final RME strategy, using questions involving computations before formal techniques for those computations have been taught, is consistent with the sociocultural principle of presenting material above what children are able to do independently but providing sufficient support so that they can accomplish the task with others prior to independent performance - i.e. working in the zone of proximal development (Vygotsky, 1978; Bruner, 1996).

State-wide reform of an upper post-primary mathematics assessment system

In Victoria, Australia a study of the impact of alternative mathematics assessment in the state-level Victorian Certificate of Education (VCE) demonstrated that there was a positive backwash effect on teaching in Years 7-10 after changes were made in assessment practices in Years 11 and 12. Year 11 and 12 assessments comprise four components: (a) multiple-choice skills test, (b) an extended answer analytic test, (c) a 10-hour 'Challenging Problem', and (d) a 20-hour 'Investigative Project' (Clarke and Stephens, 1996). These assessment tasks are undertaken in the context of a curriculum in Victoria which involves three types of formal work requirements for all students (Barnes, Clarke and Stephens, 2000):

> ***problem-solving and modelling:*** *the creative application of mathematical knowledge and skills to solve problems in unfamiliar situations, including reallife situations*
>
> ***skills practice and standard applications:*** *the study of aspects of the existing body of mathematical knowledge through learning and practising mathematical algorithms, routines and techniques, and using them to find solutions to standard problems*
>
> ***projects:*** *extended, independent investigations involving the use of mathematics. (p. 630)*

Clarke and Stephens' study, based on document analysis and the administration of questionnaires and interviews to teachers, demonstrated a strong positive ripple effect in which changes in Year 11 and 12 assessment practices were reshaping teaching and learning lower down the school system. Finally, it should be noticed that the third and fourth assessment tasks, the 10-hour 'Challenging Problem' and the 20-hour 'Investigative Project', are consistent with socio-cultural and cognitive perspectives on learning given their emphasis on extended performance assessments rooted in realworld contexts.

The Australian experience is informative in a number of ways: (i) the ripple effect on teaching and assessment practice; (ii) the pressure to scale back the 1990s assessment system due to workload and verification problems; and (iii) the continued use of problem-solving and focus on non-routine in the scaled back assessment since 2000. Firstly the 'ripple effect'; that is, how making a change in assessments can affect curriculum in terms of teachers' classroom practice. A four-year study comparing assessment practices in upper post-primary mathematics in Victoria and New South Wales (Barnes, Clarke, and Stephens, 2000) provides strong evidence that changes in assessment leveraged change in curriculum. Barnes *et al*. (2000) note that they 'sought to examine empirically the prevailing assumption that changing assessment can leverage curricular reform' (p. 623). They demonstrate that there was indeed 'congruence between mandated assessment and school wide instructional practice' and that this was found in the case of 'two states whose high-stakes assessment embodied quite contrasting values' (p. 623). Barnes *et al*. note that the support provided to teachers in terms of professional development and guidance in relation to assessment was an important factor in influencing the degree to which changing the assessments in Victoria in the early 1990s had a 'ripple effect' on classroom practices:

> *…these assessments exercise a significant leverage on teaching and forms of assessment especially when the assessment tasks are set by the examining body with extensive guidance for teachers in applying criterion-based assessment. (p. 645)*

Secondly the four-part assessment system outlined above was scaled back in 2000 and Victoria now uses a two-part system. The reasons for the change were that there were serious concerns about the authentication and verification of student work, and teacher workloads. As Brew, Tobias and Leigh-Lancaster (2001) note:

> *Following the 1997 review of the Victorian Certificate of Education (VCE) the school-based, but centrally set and externally reviewed Common Assessment Task (CAT) was discontinued in the revised VCE 2000. This action was taken in response to real and perceived problems associated with excessive student and teacher workloads and authentication of student work. (p. 98)*

Thirdly despite the scaled-back assessment system, students and teachers are still engaging in problem-solving, modelling approaches and addressing non-routine problems, partly as a legacy of familiarity with such tasks during the earlier 1990s VCE, and partly as a result of the emphasis on such tasks in the revised curriculum. As Brew, Tobias and Leigh-Lancaster note:

> *In Victoria the implementation of the new school coursework structure using application tasks, analysis tasks and tests, has been supported by the publication of considerable resources by the former Board of Studies and the Victorian Curriculum and Assessment Authority (VCAA) to encourage teachers to continue to include a variety of contexts for application tasks and different types of analysis tasks in the new school-based assessment. Within this current structure teachers are able to draw on ideas and approaches from previous extended investigative and problem solving CATs. Brew et al. (2000) provided evidence that in the first year of implementation of the revised VCE, investigations, problem-solving and modelling approaches continued to be an important component of the school-based assessment in many Victorian schools. In part, this arises from the nature of the outcomes for the revised VCE Mathematics courses that require students to apply mathematical processes in non-routine contexts. (p. 99)*

NCTM-inspired reforms in classroom assessment in the US

In the US, the National Council of Teachers of Mathematics (NCTM) has made a strong case over the last fifteen years for a range of assessment tools that will provide the kind of rich data needed to understand the ambitious teaching-for-understanding stance it has been espousing (NCTM, 1989; NCTM, 1995). NCTM has argued that assessment should be an integral part of the learning process, and that students should have opportunities to 'express mathematical ideas by speaking, writing, demonstrating, and depicting them visually' (NCTM, 1989, p. 14). The NCTM's position on assessment reflects a movement internationally in which 'mathematics educators are moving from a view of mathematics as a fixed and unchanging collection of facts and skills to an emphasis on the importance in mathematics learning of conjecturing, communicating, problem-solving and logical reasoning' (Lester, Lambdin and Preston, 1997, p. 292). Lester *et al*. (1997) identify three key shifts in thinking that have occurred in mathematics education, encompassing the nature of mathematics, the nature of mathematics learning, and the nature of mathematics teaching. These in turn have promoted changes in mathematics assessment in the US (see Table 4).

Table 4: Changes in views of mathematics, mathematics learning and mathematics teaching.

Assumptions	Traditional view	Modern view
The nature of mathematics	Mathematics is nothing more than a list of mechanistic condition/ action rules	Mathematics is a science of patterns
The nature of mathematics learning	Mathematics learning is a cumulative process of gradually adding, deleting, and debugging facts, rules and skills	Humans are model builders, theory builders; they construct their knowledge to describe, explain, create, modify, adapt, predict, and control complex systems in the real world (real or possible)
The nature of mathematics teaching	Teaching involves demonstrating, monitoring student activity, and correcting errors	Teaching is an act of enabling students to construct and explore complex systems

They also note that a new generation of ICTs such as 'handheld calculators and notebook computers with graphing and symbol manipulation capabilities enables students to think differently, not just faster' (p. 293), and argue that, in this new landscape, teachers have to determine the optimal uses of technology to enhance learning and then decide on how best to assess the types of mathematical reasoning made possible by ICTs. Approaches to alternative assessments in mathematics education include the use of portfolios, assessments of cooperative group work, use of more structured observations of students, interviews with students, open-ended and extended tasks (see Victoria, Australia), concept maps, revision of student work (this was the strategy used in the Connected Mathematics Project), and student journals. Scoring of these various alternative assessments has adopted one of three strategies: general impression, analytic or holistic scoring. In using general impression scoring, teachers apply their past experience to gauge student work and award a score/mark rather than relying on specified criteria. In analytic scoring, the most time-consuming, teachers focus on the components of, for example, a problem-solving task and rate students' response on pre-specified criteria for each component of the problem. In the case of holistic scoring, a pre-specified rubric or set of criteria are used to provide an appraisal of students' whole response to a unit of work or problem-solving task.

In addition to creating a context for new approaches to assessment, the NCTM identified six standards for judging the quality of mathematics assessments: mathematics, learning, equity, openness, inferences and coherence.

- Standard 1 focuses on mathematics in terms of the value of choosing worthwhile mathematical ideas and by implication raises questions about the atomistic content of many traditional standardised mathematics tests with their focus on isolated bits of knowledge.

- Standard 2 focuses on learning in order to ensure that assessments are embedded in the curriculum and inform teaching.

- Standard 3 focuses on equity and opportunity and emphasises that assessments should give every student an opportunity to demonstrate mathematical competence.

- Standard 4 focuses on openness and notes that traditionally testing and examinations have been a very secretive process where test questions, sample answers and criteria for assessing responses have been closely guarded by unidentified authorities. NCTM argues for a more open approach where criteria and scoring procedures in particular are made public.

- Standard 5 focuses on inferences and redefining the traditional psychometric testing principles of reliability and validity given new curriculum-embedded uses of assessment. In the case of reliability, it makes little sense to think of new assessments being reliable across test instances (i.e. test-retest reliability, one of the traditional psychometric conceptions of reliability) when the goal is to use these assessments to assess change in learners' understanding. In the case of validity, NCTM makes a case for moving away from thinking of it in terms of an inherent feature of the test itself towards a view of validity as the quality of inferences made from the test/assessment.

- Standard 6 focuses on coherence and stresses the 'goodness of fit' between each type of assessment instrument in terms of the purposes for which it is used.

Many authors make cogent arguments for alternative assessment in mathematics education (e.g. Lester *et al.*, 1997). Despite the many advantages of alternative assessments, Lester *et al.* (1997) also identify problems with their implementation including time constraints, monetary costs, teachers' limited knowledge of alternative assessments, and difficulties in creating authentic tasks. We might add another problem: the challenge of creating alternative assessments which are fair, as some research demonstrates, according to Elwood and Carlisle (2003), that there are gender differences in how students respond to more realistic-oriented assessment items. Furthermore, Cooper and Dunne (2000) have argued that there are significant differences in how students from different socio-economic groups respond to realistic-oriented assessments, with lower SES students less likely to draw upon relevant knowledge. In the USA, the NCTM view of mathematics reform was influenced by developments in cognitive and learning sciences, rather than a domain-specific view of mathematics as in the highly influential Realistic Mathematics Education movement which originated in the Netherlands.

3.3 Realistic Mathematics Education (RME) and learning

This section of the chapter examines the work of Hans Freudenthal and RME, focusing on four aspects: the roots of RME, big ideas in RME, the legacy of RME, and challenges in living out RME in practice.

RME's roots

> *During his professional life, Hans Freudenthal's views contradicted almost every contemporary approach to educational reform: the 'new' mathematics, operationalized objectives, rigid forms of assessment, standardized quantitative empirical research, a strict division of labour between*

> *curriculum research and development, or between development and implementation. (Gravemeijer and Terwel, 2000, p. 777)*

RME has its roots in the work of Hans Freudenthal, a mathematician turned mathematics educator, highly critical of mainstream mathematics education from the 1950s given its 'instructional design' emphasis and related hierarchical assumptions, its basis in Bloom's *Taxonomy*, and its measurement focus (see Freudenthal's book *Weeding and Sowing*, 1980). Freudenthal died in the early 1990s but his work and ideas are being developed through research undertaken in the Freudenthal Institute in the Netherlands, as well as by an increasing number of researchers using RME to inform their own work (e.g. Cobb *et al.*, 1997; Romberg, 2000). Good primary source overviews of RME are available in Freudenthal's last book *Revisiting Mathematics Education: China Lectures* (1991) and a collection of articles published by his colleagues in 1993 and reprinted in 2001 titled *The Legacy of Hans Freudenthal* (Streefland, 2001).

The big ideas in RME

Freudenthal's view of maths education cannot be appreciated without understanding his objections and resistance to the formal and abstract ideas at the core of the new or mathematics movement. While new mathematics elevated abstraction as its highest value, Freudenthal saw this as its primary weakness, stating that, 'In an objective sense the most abstract mathematics is without doubt also the most flexible. But not subjectively, since it is wasted on individuals who are not able to avail themselves of this flexibility' (Freudenthal, 1968, p. 5). Rather, based on his own experiences as a mathematician, he viewed mathematics as a human activity deeply embedded in real situations. Furthermore, his understanding of the growth of mathematics, as a discipline or set of related areas of human inquiry, led him to stress its social and cultural embeddedness. For the purposes of this report, we focus on three ideas in RME: reality and related notions of rich pedagogical contexts; the horizontal and vertical mathematising cycle; and the four-level framework for classifying curricular emphases in mathematics education.

Firstly Freudenthal's conception of reality is central to his espoused pedagogy, as it is from a basis in some real situation that learning mathematics has its source and it is also the context for the application of formal mathematical ideas. This is a very important reconceptualisation of the role of reality in mathematics education, as reality is typically seen only as the site where mathematical models are applied, rather than as a rich source of mathematical ideas. While Freudenthal objected strongly to constructivism, viewing it no more than empty sloganising, his own conception of reality has a distinctly constructivist feel. Freudenthal's phenomenological conception of reality and its emphasis on the learner's perspective, is constructivist in the sense that he recognised that particular experiences have the potential to be more real - or to mean something different - from one learner to another. For example, he argues that: 'Real is not intended to be understood ontologically… but instead commensically… it is not bound to space-time world. It includes mental objects and mental activities. What I called "expanding reality" is accounted for on ever higher levels of common sense and witnessed by levels of everyday language or various technical languages' (p. 17).

Secondly the notion of mathematising or mathematical modelling is central to RME. Initially proposed by Treffers (1987), Freudenthal was reticent to endorse it but came to see its value in helping to formalise key ideas in RME (Freudenthal, 1991, pp. 41-44). After he agreed on the value of the distinction, Freudenthal described it as follows:

> *Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is a symbol world with regard to abstraction. To be sure, the frontiers of these worlds are vaguely marked. The worlds can expand and shrink also at one another' s expense. (1991, p. 71)*

Based on this definition of mathematising, new mathematics stresses - and indeed almost exclusively confines mathematising to - the vertical dimension. The vertical-horizontal distinction is evident in the PISA mathematical literacy framework which we discuss later in this chapter (see Section 3.5, Figure 2).

Thirdly Freudenthal provides a four-level framework for classifying curricular emphases in mathematics (1991, pp. 133-37). This is a valuable heuristic in appraising the policy direction to be taken, or not taken, in efforts to change mathematics education (Table 5).

Table 5: Curricular emphases on horizontal and vertical mathematising

	Horizontal	Vertical
Mechanistic	-	-
Empiricist	+	-
Structuralist	-	+
Realistic	+	+

Furthermore, this four-level framework is useful in considering the extent to which different levels of mathematics syllabi (e.g. Foundation, Ordinary and Higher) provide all learners with opportunities to experience and engage comprehensively with the full cycle of mathematising, including both horizontal and vertical aspects of mathematical modelling. Freudenthal's emphasis on the importance of mathematising reality as the basis for all mathematics education is consistent, as Gravemeijer and Terwel (2000) stress, with the principle of 'mathematics for all'. Freudenthal delineates four possible types of curricular stance in mathematics: mechanistic, empiricist, structuralist and realistic. Each is defined by the presence or absence of vertical and/or horizontal mathematising. Mechanistic mathematics employs neither horizontal nor vertical mathematising, but is focused on routine mathematical drills. The empiricist approach emphasises horizontal mathematising, working mathematically real life situations to symbols without delving into the world of symbolic manipulation. Structuralist mathematics, consistent with 'new' mathematics, focuses on symbolic manipulation, that is, vertical mathematising. Realistic mathematising combines both the horizontal and vertical. In doing so, the realistic approach can encompass the dual nature of mathematics, as described by De Corte *et al.* (1996):

> *On the one hand, mathematics is rooted in the perception and description of the ordering of events in time and the arrangement of objects in space, and so on ('common sense - only better organised', as Freudenthal (1991, p. 9) put it), and in the solution of practical problems. On the other hand, out of this activity emerge symbolically represented structures that can become objects of reflection and elaboration, independent of their own real-world roots. (p. 500)*

In the Irish situation, one of the dangers in reforming mathematics within a system where there are tiered mathematics syllabi (e.g. Foundation, Ordinary and Higher level) and/or different types of mathematics syllabi (e.g. pure versus applied mathematics) is that learners will be deemed suitable for only a portion of the full mathematising cycle, thus encompassing only one aspect of the dual nature of mathematics. For example, more advanced mathematics students might be seen as more suited to vertical mathematising and thereby miss out the value of moving between the vertical and horizontal. On the other hand, a more horizontally-oriented syllabus might be designed for less advanced students, thereby excluding them from the value of vertical mathematics. We want to emphasise that there is no case being made in Freudenthal's writings or in the wider RME research community that vertical and horizontal mathematising are more suited to particular learners, either in terms of age, developmental level or cultural background. Rather, the focus in RME is on mathematics for all and the value of providing all students with experiences of the full mathematising cycle in the context of mathematics as a human activity. Thus, from an RME perspective, all syllabi, whether at Foundation, Ordinary or Higher level, should provide many opportunities for students to experience horizontal and vertical mathematising and explore their inter-relationship.

The legacy

The Freudenthal Institute, the impact of RME on conceptions of maths education including textbook development, the Netherlands' high rankings in PISA, the ongoing research in RME tradition, and the impact of RME on mathematics education, most notably its adoption by PISA in the context of developing a framework for mathematics literacy, are the lasting legacy of Freudenthal. As such, the work of Freudenthal represents a huge contribution to mathematics education. Freudenthal's ideas were marginal when 'new' mathematics was dominant but the tide has turned, in part due to the emergence of new understandings of learning from research on situated cognition, as well as the widely perceived need for a more socially embedded mathematics education in order to prepare students to think mathematically and apply knowledge in new contexts. Furthermore, the adoption of horizontal (real world to symbol) and vertical (symbol to symbol) mathematising represents a powerful model for framing the scope of mathematics in planning and enacting curriculum, giving due regard to student experience, the complexity of learning from - and applying mathematics in - real world contexts, and the powerful conceptual traditions of various branches within mathematics.

The challenges: how realistic is real-world maths?

Perhaps the main challenge of implementing RME is coming to grips with the meanings of 'realistic' and 'context'. There are other challenges, including demands on teachers' time and understanding of maths, access to resources, and the manner in which historically dominant curricular cultures distort the meaning of RME through the subtle influence of alternative perspectives embedded in textbooks and tests/examinations. But the biggest challenge is defining what counts as a rich context for mathematising.

A move towards more applications-focused mathematics is one of ways in which mathematics educators have been attempting to make mathematics more realistic (Cooper and Harries, 2002). Textual representation of mathematics problems as word stories is the primary means of achieving this goal. But although presenting problems in the form of stories is the traditional way of making maths real, word story problems are problematic. As numerous studies internationally demonstrate, students appear to engage in widespread 'suspension of sense-making' when presented with word story problems, though it is a centuries-old method of making mathematics real for students. For example, research in various education systems (e.g. Belgium, Northern Ireland, Germany and Switzerland) reveals a similar pattern of response to word story problems, whereby students tend to approach them in such a way that they marginalise their out-of-school knowledge.

We illustrate the challenges of creating more authentic school mathematics experiences by drawing on the mathematical word story problem research of Verschaffel, Greer and De Corte (2000). For example, they posed the following three word story problems to primary and post-primary students in different countries over a number of years:

> *450 soldiers must be bussed to an army barracks. Each bus holds 36 soldiers. How many buses are needed? (Bus problem)*
>
> *Steve has bought four planks, each 2.5m. How many planks 1m long can he saw from these planks? (Plank problem)*
>
> *Joan's best 100m time is 17 seconds. How long will it take her to run 1km? (Running problem)*

As Cooper and Harries (2002) note, students may respond 'realistically' in two ways. First, knowing the predictable and solvable format of school-sourced mathematics story problems in which the story loosely disguises a set of numbers, students may discard any real-world considerations in developing a mathematical model. So, for example, in the running problem, they may ignore the question of whether Joan will be able to maintain the 17-second per 100m rate for a full kilometre. Such lack of realism in many school-sourced word story problems has accumulated in students' mathematics learning histories and results in a marginalisation of students' real-world knowledge. Second, students may draw upon their everyday out-of-school experience. In terms of the goals of promoting 'realistic' mathematics, in constructivist-and RME-inspired curricula, the hope is that they will.

How do students typically respond? Based on either written or oral responses, only 49% of students in the case of the bus problem, 13% in the case of the plank problem and 3% in the case of the running problem responded in ways hoped for from a 'realistic' mathematic perspective. Not surprisingly, students provide answers which indicate that they respond by accepting the logic implied in the genre of school word story problems. They typically assume textbook problems are solvable and make sense, that there is only one correct answer, that the answer is numerical and precise, that the final result involves 'clean' numbers (i.e. only whole numbers), and that the problem as written contains all the information needed to solve it (Verschaffel, Greer and De Corte, 2000, p. 59).

Verschaffel *et al*. took another version of the bus problem:

> *328 senior citizens are going on a trip. A bus can seat 40 people. How many buses are needed so that all the senior citizens can go on the trip?*

They presented this problem and a similar one in two ways to the same group of students in an effort to understand problem authenticity. They first presented the problem as above, and second they placed the students in a more 'reality-based situation'. In the 'reality-based situation', students were asked to make a telephone call using a tele-trainer from a local telemarketing company and were given the following information:

> ***Facts***
>
> *Date of party: Fri. April 15 Time: 4-6pm*
>
> *Place: Vinnie's Restaurant, Queen's*
>
> *Number of children attending party: 32*
>
> *Problem: We need to transport the 32 children to the restaurant so we need transportation. We have to order minivans. Board of Education minivans seat 5 children. These minivans have 5 seats with seatbelts and are prohibited by law to seat more than five children. How many minivans do we need? Once we know how many minivans are needed, call 998-2323 to place the order.*

In the first (restrictive) presentation, only 2 of the 20 students responded appropriately; the other 18 gave an incorrect interpretation of the remainder. In the 'reality-based situation' 16 of the 20 students gave an appropriate response; that is, they justified their answer for using the number of minivans (e.g. 6 minivans plus a car for the remaining 2 students).

Commenting on the 'unrealistic' nature of most school mathematics textbook problems, Freudenthal noted that:

> *In the textbook context each problem has one and only one solution: There is no access for reality, with its unsolvable and multiply solvable problems. The pupil is supposed to discover the pseudo-isomorphisms envisaged by the textbook author and to solve problems, which look as though they were tied to reality, by means of these pseudo-isomorphisms.*

> *Wouldn't it be worthwhile investigating whether and how this didactic breeds an anti-mathematical attitude and why children's immunity against this mental deformation is so varied? (1991, p. 70)*

The work of Verschaffel *et al*. (2000), Cooper and Harries (2002) and others questioning how schools purport to present 'realistic' mathematics provides a challenge for mathematics educators and curriculum designers alike as they seek to create more 'realistic' mathematics education, whether they are motivated by constructivist, situated cognition or RME perspectives on mathematics.

3.4 Situated cognition in mathematics education

Another important voice in this discussion is represented by situated cognition. Much of the research conducted from a situated perspective has demonstrated how students utilise particular mathematical strategies, not solely because of some cognitive level of achievement, but in part because of the socio-cultural context: the nature of the activity, working practices, features of the tasks, etc. Many of the theoretical insights in situated cognition grew out of ethnographic studies of everyday mathematics in out-of-school contexts, and were undertaken by cognitive anthropologists and cultural psychologists (Lave, 1988; Walkerdine, 1988; Brown, Collins and Duguid, 1989; for reviews see de Abreu, 2002; Saxe, 1999; Saxe and Esmond, 2005). For example, in section 3.1 of this chapter, we noted research in this tradition undertaken by Nuñes, Schleimann and Carraher (1993) which demonstrated how street children in Brazil working as street vendors had developed complex algorithms to calculate price and quantity in a fast and accurate fashion. The same children when presented with similar tasks in symbolic form in school situations performed well below their on-street level.

Collectively such research has demonstrated that children and adults often have robust expertise in out-of-school settings.

As a further example, Schliemann, Goodrow and Lara-Roth (2001), in a study of third-grade children, used Vergnaud's (1983) scalar versus functional distinction to characterise the relative merits of different problem-solving strategies:

> *In our earlier work (Nuñes, Schliemann, and Carraher, 1993), we found that street sellers compute the price of a certain amount of items by performing successive additions of the price of one item, as many times as the number of items to be sold. The following solution by a coconut vendor to determine the price of 10 coconuts at 35 cruzeiros each exemplifies this:*
>
> *"Three will be one hundred and five; with three more, that will be two hundred and ten. [Pause]. I need four more. That is…[Pause] three hundred and fifteen…I think it is three hundred and fifty."*
>
> **(Nuñes, Schliemann, and Carraher, 1993, p. 19).**
>
> *The street sellers perform operations on measures of like nature, summing money with money, items with items, thus using a scalar approach (Vergnaud, 1983). In contrast, a functional approach relies upon relationships between variables and on how one variable changes as a function of the other. While work with scalar solutions can constitute a meaningful first step towards understanding number or quantity, a focus on scalar solutions does not allow for broader exploration of the relationships between two variables. Schools should therefore provide children with opportunities to explore functional relationships.*

In conceptual analyses or cognitive models, *functional* strategies are typically conceived as more sophisticated than *scalar* strategies.

However, in work and everyday situations, practitioners rely more heavily on scalar strategies and often employ them even when a functional strategy would be easier computationally. In part this is because the scalar strategies allow people to keep track of the measurable attributes of quantities in situations (e.g., dollars, pounds, feet), and thus preserve their ability to engage in sense-making in situations. That is, people can work with quantities and relationships and not just numbers stripped of contextual details when working with scalar strategies. Similarly, Säljö and Wyndhamn (1996) demonstrate how significantly students' maths strategies change when the same task is posed in a mathematics class as against a social studies class. In short, it is important for researchers creating cognitive models also to account for learners' goals, the activity setting, artefacts, socio-cultural practices, and phenomenological experiences. Thus, learning that appears to be hierarchically ordered and following a single path from a behaviourist or cognitive perspective may appear to be more multi-path in nature when that learning is viewed from a situated perspective.

We believe that one must consider conceptual change in mathematics as both a shared characteristic and as an individual psychological phenomenon. Students learn mathematics as they participate in communities of practice (Cobb and Yackel, 1996) and engage in both individual and social processes of learning. While one could consider social interaction simply as a catalyst for individual psychological development, other researchers have criticised this view for not acknowledging that students' interpretations of events as interpersonal conflict are in fact influenced by the classroom practices in which they participate (Salomon, 1993). Instead, we contend that cognition should be viewed as inherently social and as distributed across individuals as well as occurring within individuals.

A well-rounded study of mathematics learning should take into account not only interactions with materials and the cultural context of activity, but also the contribution of social interactions. Thus we consider the work of social theorists (Bowers, Cobb and McClain, 1999; Cobb, McClain and Gravemeijer, 2003; McClain and Cobb, 2001; Sfard and Kieran, 2001; Yackel, Cobb and Wood, 1999) to be critical in helping the mathematics education community to better understand the distributed, social nature of learning.

This expanded view of learning can explain some phenomena not accounted for within a purely psychological perspective. For example, it allows researchers to: (i) examine the ways in which cognition can be partly offloaded onto the external environment via interaction with tools, artefacts and other people (Brown, Collins and Duguid, 1989; Hutchins, 1995a; Hutchins 1995b); (ii) study the structuring resources for cognition that are distributed through the personal, social and historical settings in which people live and work (Salomon, 1993; Nuñes, Schliemann and Carraher, 1993); and (iii) capture interpersonal discourse as a tool for directly analysing cognitive events (Sfard and Kieran, 2001). However, attention to situated activity as a research focus could result in lack of attention to individual understanding and development. The study of internal schemes and mental operations are typically downplayed when a social perspective guides data collection and analysis (Lobato, 2005). Cobb and Yackel (1996) cautioned that a group analysis tends to downplay qualitative differences in individual students' mathematical interpretations, except to the extent to which they can be tied to their participation in communities of practice. While tracking changes only in individuals' conceptions might leave out important data about emerging social understanding and the reflexive relationship between the two, tracking group conceptions alone could similarly result in a half-told story. In summary, situated cognition challenges in very significant ways contemporary notions of the individual learner, emphasising the central role cultural and social settings as well as various technologies play in shaping mathematical ways of knowing in or out of school. The artificial nature of school word story problems is a good example of situated cognition in that over time learners come to see their artificial nature and respond accordingly, that is, by suspending sense making. A situated cognition perspective challenges educators to think beyond individual ability as the sole indicator of mathematical competence, and puts an emphasis on viewing mathematics within the context of activities, language, and social and educational expectations in or out of school. Finally, a situated cognition perspective challenges the education system to consider ways in which it can leverage social, cultural and technological resources to equip students with skills and knowledge to engage with mathematics both in and out of school.

3.5 The PISA mathematics literacy framework: situated cognition and RME

We now turn to how PISA addresses the challenges posed by situated cognition and RME. The PISA framework is based on the assumption that mathematics is a human activity, and reflects the strong influence of both RME and situated cognition. We can use an example to illustrate the PISA mathematical literacy framework (source: OECD, 2003, Mathematics Example 1, pp. 26-27).

Mathematics Example 1: *Streetlight*

The Town Council has decided to construct a streetlight in a small triangular park so that it illuminates the whole park. Where should it be placed?

According to the PISA Framework (OECD, 2003), the above social problem 'can be solved by following the general strategy used by mathematicians, which the mathematics framework will refer to as mathematising. Mathematising can be characterised as having five aspects' (p. 26):

1. Starting with a problem situated in reality.
   Locating where a streetlight is to be placed in a park.

2. Organising it according to mathematical concepts.
   The park can be represented as a triangle, and illumination from a light as a circle with the street light as its centre.

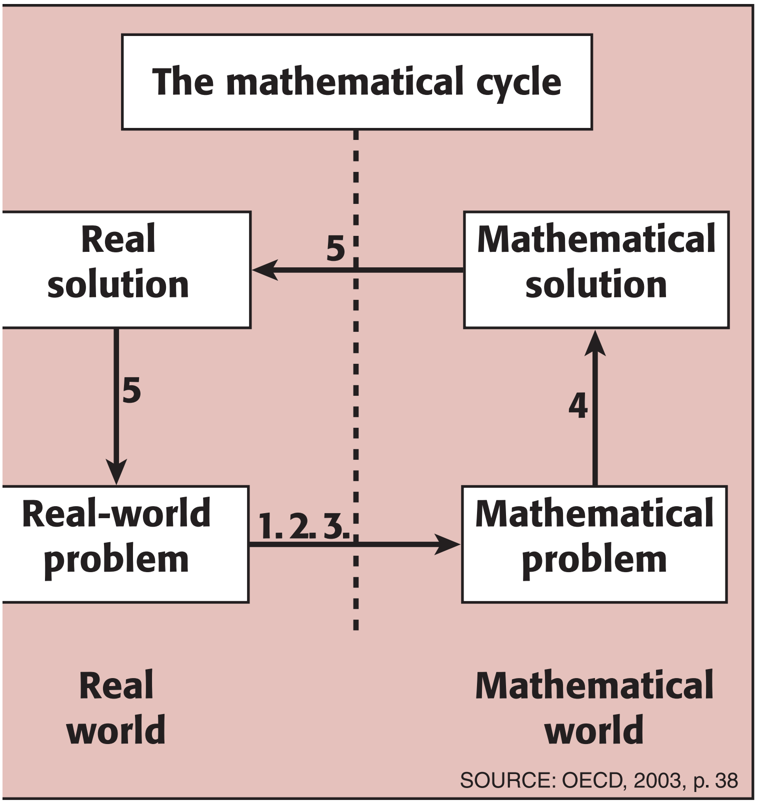
3. Gradually trimming away the reality through processes such as making assumptions about which features of the problem are important, generalising and formalising (which promote the mathematical features of the situation and transform the real problem into a mathematical problem that faithfully represents the situation).
   The problem is transformed into locating the centre of a circle that circumscribes the triangle.

4. Solving the mathematical problem.
   Using the fact that the centre of a circle that circumscribes a triangle lies at the point of intersection of the perpendicular bisectors of the triangle's sides, construct the perpendicular bisectors of two sides of the triangle. The point of intersection of the bisectors is the centre of the circle.

5. Making sense of the mathematical solution in terms of the real situation. *Relating this finding to the real park. Reflecting on this solution and recognising, for example, that if one of the three corners of the park were an obtuse angle, this solution would not be reasonable since*

the location of the light would be outside the park. Recognising that the location and size of trees in the park are other factors affecting the usefulness of the mathematical solution.

Figure 2: The mathematical cycle



The five phases in this mathematising cycle can be represented diagrammatically (see Figure 2) From a new/modern mathematics perspective, the mathematical 'action' is almost all on the right hand side of the cycle, that is, in the mathematical world. In the language of RME, it corresponds to a focus on vertical mathematics. However, providing students with opportunities to experience the full mathematising cycle, as a routine part of classroom mathematics culture, is vital in the promotion of mathematics in context.

1. Starting with a problem situation in reality.

2. Organising it according to mathematical concepts and identifying the relevant mathematics.

3. Gradually trimming away the reality through processes such as making assumptions, generalising and formalising, which promote the mathematical features of the situation and transform the real-world problem into a mathematical problem that faithfully represents the situation.

4. Solving the mathematical problem.

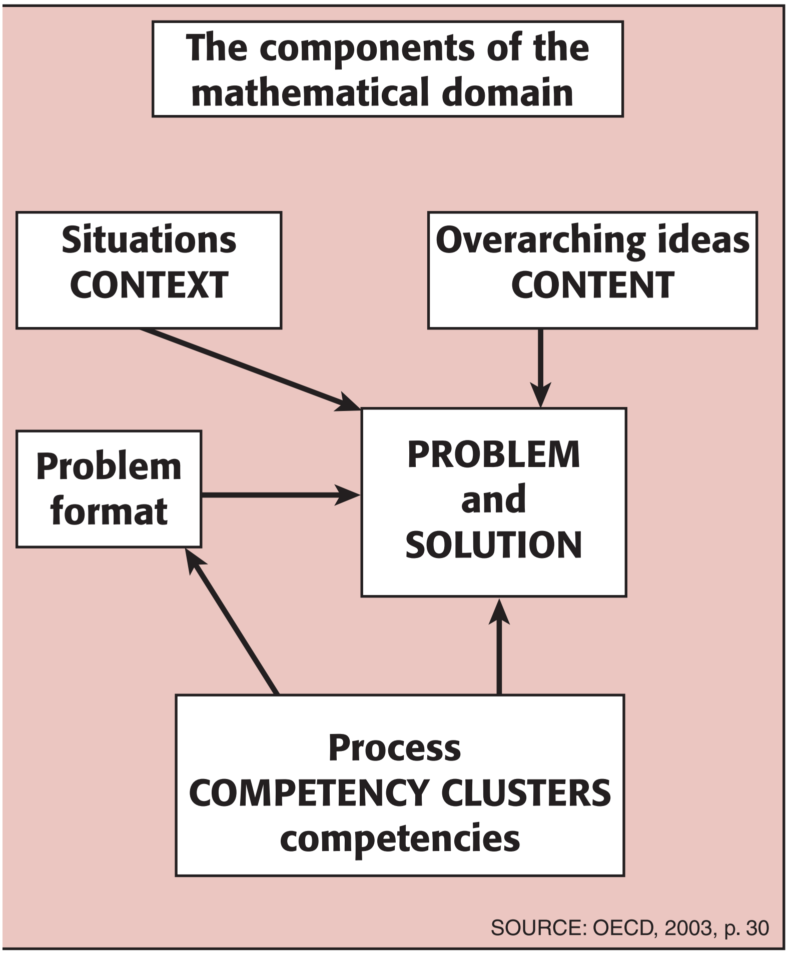
5. Making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution.

Typical of RME application problems, the PISA mathematical literacy items focus on application problems rather than traditional word story problems. The problem is rooted in the real world and students are expected to traverse both horizontal and vertical mathematising in coming to a solution. The vertical-horizontal boundary is one that students are expected to be able to move seamlessly across as they consider they validity of any chosen mathematical models and then reflect on any solutions in terms of the real world. Thus, if the real and mathematical worlds are opposite banks of a river, students are expected to cross and re-cross the river, viewing one side from the perspective of the other.

The components in the PISA mathematical domain

After starting with an example of a 'real-world' PISA item, we now turn to the components in the PISA mathematical domain made up of competency clusters, contexts (situations), and content (overarching ideas). Competency clusters reflect the different cognitive challenges intended in various types of problems. They are defined and discussed above in the section on the cognitive approach to learning (Section 3.1). In contrast with, for example, behaviourally-based assessment items, PISA assessment items reflect an assumption that various mathematical processes and strategies will be used concurrently: 'The process of mathematics as defined by general mathematical competencies… include the use of mathematical language, modelling and problem-solving skills.

Figure 3: The components of the mathematical domain



Such skills, however, are not separated out in different test items, since it is assumed that a range of competencies will be needed to perform any given mathematical task' (OECD, 2003, p. 16). Based on the assumption, drawn from both RME and situated cognition, that mathematical literacy involves engaging with mathematics in different situations, PISA specifies four contexts in which mathematics can be presented to students: personal, educational/ occupational, public and scientific (OECD, 2003, p. 32). These situations are defined 'based on their distance to the students' (OECD, 2003, p. 16), that is students' experience of everyday life.

Finally, mathematical literacy content is conceptualised in terms of four overarching ideas:

- quantity
- space and shape
- change and relationships
- uncertainty.

The organisation according to overarching ideas contrasts with organisation by curricular strand typical of many curriculum documents, as explained in the PISA Framework document, 'the mathematical content [is] defined mainly in terms of four "overarching ideas" (quantity, space and shape, change and relationships, and uncertainty) and only secondarily in relation to "curricular strands" (such as number, algebra and geometry)' (p. 15).

3.6 Neuroscience as a basis for mathematics education: is it a bridge too far?

This section provides an overview of research into neuroscience and debates about the utility of such knowledge as a conceptual lens on the teaching and learning of mathematics. There is considerable interest internationally in the potential of brain-based research to inform mathematics education. The OECD's recent publication *Understanding the Brain: Towards a New Learning Science* (2002) details recent neuroscience research outlining its implications for, and the conditions for its use in, education. The OECD's interest in brain-based research is indicative of an international movement directed at understanding the potential of recent rapid advances in what is known about the brain, how it operates, and how it can be influenced, largely thanks to research based on sophisticated brain scanning technologies such as positron emission tomography (PET) and functional magnetic resonance imaging (fMRI). These technologies have shed light on normal neurobiological development as well as developmental disabilities (e.g. dyslexia). Brain-based research has attracted the attention of some educational policy makers. For example, one of Asia-Pacific Economic Cooperation's (APEC, 2004) five proposed priority action steps, in relation to APEC-wide collaboration to stimulate learning in mathematics and science, is a focus on brain-based research:

> *APEC should work to determine how brain research applies to the teaching of mathematics and science concepts, in determining appropriate sequencing of concepts, and in helping students of different ages to retain mathematical and science concepts. APEC should also consider information collected through UNESCO and OECD brain research projects. (p. 63)*

In this section, we examine some key insights from neuroscience research. We draw on two key reports, the US National Research Council (NRC) report *How People Learn* (2000) and the OECD's *Understanding the Brain: Towards a New Learning Science* (2002). We conclude the section by assessing whether brain-based research might guide development of policies and practices in the teaching and learning of mathematics. As such, we focus on whether current brain-based research knowledge can extend to form a bridge with the practice of mathematics education, or whether this is a bridge too far (Bruer, 1997).

The argument in favour of a brain-based approach to education, according to Bruer (1997), rests on 'three important and reasonably well established findings in developmental neurobiology' (p. 4). The first of these notes that from early infancy until middle childhood there is a period of rapid brain development characterised by the proliferation of synapses, then followed by a later period of pruning or elimination. Secondly there are experience-dependent critical periods in sensory and motor development. Thirdly considerable research over the last thirty-plus years has demonstrated the positive effects of experience-rich environments on rats' brain development, and by extrapolation there is a strong case for the occurrence of similar process in humans (Bruer, 1997). The NRC Report summarises its main points in relation to the current state of knowledge on learning from the field of neuroscience as follows:

- Learning changes the physical structure of the brain.
- These structural changes alter the functional organisation of the brain; in other words, learning organises and reorganises the brain.
- Different parts of the brain may be ready to learn at different times.

In order to outline the manner in which learning changes the physical structure of the brain, it is important to identify two components of the brain important in development: the neurons, or nerve cells, and synapses, the brain's information junctions. Two mechanisms account for changes in synapse development. Over the course of the first ten years of life, synapses are overproduced and pruned or selectively lost. A second mechanism involving the addition or growth of synapses is due to the nature of an individual's experiences. This second process is 'actually driven by experience' and forms the basis for memory (Bransford, Brown and Cocking, 2000). Studies comparing the brains of animals raised in complex environments with those of animals raised in environments lacking stimulation, provide convincing evidence that enriched experiences result in an 'orchestrated pattern of increased capacity in the brain that depends on experience' (Bransford, Brown and Cocking, 2000, p. 119). Such orchestrated changes occur in localised brain areas depending on the type and quality of experiences. Furthermore, synapse loss and growth occur at different rates in different parts of the brain depending both on learning experiences and internal development processes.

In what ways - and to what extent - might brain-based research inform mathematics education? Firstly brain research has demonstrated that different types of experiences have different effects on the brain. Secondly at present there is not sufficient evidence from a neuroscientific perspective to recommend certain activities as pedagogical strategies supported by brain-based research. As such, the exact implications for classroom practice are not at a level where specific practices can be recommended to ensure neural branching (Bransford, Brown and Cocking, 2000). Thirdly a critical approach to recommendations from various quarters, citing brain-based research for their educational recommendations, is merited, as these are often based on tenuous scientific evidence. Finally, it is important that teachers are aware of the insights and current limitations of brain-based research in informing pedagogical practice in mathematics.

3.7 Fostering ownership of learning: learning to learn

> *…the development of education and training systems in a lifelong learning and in a worldwide perspective has increasingly been acknowledged as a crucial factor for the future of Europe in the knowledge era.*
>
> **(Detailed work programme on the follow up of the objectives of education and training systems in Europe, European Union, 2000, p. 9)**

> *Teaching students self-regulatory skills in addition to classical subject-matter knowledge is currently viewed as one of the major goals of education. At the same time, self-regulated learning (SRL) is a vital prerequisite for the successful acquisition of knowledge in school and beyond, and is thus of particular importance with respect to lifelong learning.*
>
> **(Baumert *et al.*, 'Self-regulated learning as a cross-curricular competence', OECD-PISA Consortium, 1999, p. 1)**

'It was drilled into me' as an expression of a learner's relationship to learning represents the antithesis of the idea of promoting ownership of learning. Whereas the learner's sense that knowledge was drilled into him or her leaves the learner in a passive or receptive state, the promotion of a learning to learn capacity seeks to enlist the learner in his or her own education. For example, in mathematics problem-solving many learners assume that if after five minutes a problem has not been solved it is not possible to solve it, and therefore they typically give up (Greeno, Pearson and Schoenfeld, 1997). Furthermore Greeno, Pearson and Schoenfeld (1997) note that, in many fields, people think that 'you either know it or you don't' and therefore give up prematurely in the face of difficult problems.

A recurrent focus on the importance of *learning to learn* in the context of promoting *lifelong learning* is one of the most distinctive features of contemporary educational and economic policy-making at national and international levels. As such, the development of a learning to learn capacity has become a key educational policy priority around the world, animating discussions about the purpose of schooling as well as debates about how learning to learn can be developed in specific subject areas. As we have noted already, it has been emphasised as a cross-curricular goal in APEC countries. Self-directed learning, capacity for independent and collaborative problem-solving, developing ownership of learning, and self-regulated learning are terms used interchangeably in terms of highlighting learning to learn as an educational aim. In the case of mathematics, the promotion of skilled problem-solving has been a long-standing concern among mathematics educators (De Corte, Greer and Verschaffel, 1996). For example, early work by Polya (1945) on problem-solving heuristics (understand the problem, find the connection between the data and the unknown by possibly considering related problems, develop a plan, carry it out, and examine the solution) and more recent work by Schoenfeld (1985, 1987, 1992) attempted to teach mathematical problem-solving long before the contemporary focus on learning to learn and lifelong learning. Schoenfeld (1992), in particular, focused on delineating heuristic strategies as well as meta-cognitive control skills.

Why is learning to learn appealing and important to policy makers? Firstly there is an acknowledgement that with the accelerating pace of social change, economic development and most importantly knowledge production, it is crucial that learners have the capacity to continue engaging with new knowledge and ideas over the course of their personal and professional lives. Secondly the current wave of globalisation presents challenges, such as the explosion of information, environmental sustainability, pandemics (e.g. HIV/AIDS, SARS), terrorism, and national and global inequalities, that demand deep disciplinary knowledge, the capacity for interdisciplinary knowledge construction, and competence in dealing with non-routine problems (Gardner, 2001). As such, future challenges that today's students will have to address will demand competence in how they can manipulate, reframe, connect and apply knowledge to ill-structured problems (within and across and disciplines) and continue to do this throughout their lives. Thirdly in an era of lifelong learning, school graduates are expected to enter the workforce, higher education, or further education with the capacity for promoting their own learning.

In addition to APEC, the OECD's PISA framework also identifies self-regulated learning as a necessary stepping-stone in the promotion of lifelong learning, which has become a cradle-to-grave educational aim in many regions of the world. Baumert *et al.* (1999), in an article conceptualising self-regulated learning in the context of the OECD PISA studies, notes (i) the many definitions of SRL, (ii) the increasing move away from a sole focus on cognitive strategies to incorporate motivation and context, and (iii) the complexity of assessing students' SRL through questionnaires which focus on SRL as a domain-general capacity (i.e. non subject-specific competence) given that considerable research has demonstrated important subject-specific aspects of SRL (Chi, 1987; De Corte, Greer and Verschaffel, 1996; Resnick, 1987; Zimmerman, 2001).

While cognisant of Baumert's observation about the proliferation of SRL definitions, for the purposes of this report we use Pintrich's definition as, we think, it captures the current understanding of self-regulated learning as a multi-dimensional construct encompassing both self-regulatory processes (planning, monitoring, control and review) and areas for self-regulation (cognition, motivation, behaviour and context) (Schunk, 2005). According to Pintrich, self-regulated learning, or self-regulation, is 'an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and constrained by their goals and the contextual features in the environment' (2000, p. 453). Pintrich's definition of SRL provides a framework for guiding the design and evaluation of SRL interventions. Schunk (2005), citing Boekaerts *et al*. (2000) and Schunk and Zimmerman (1998), notes that the design and evaluation of SRL interventions in subject areas has demonstrated that self-regulation can be taught to students, and the outcome of training studies

> *…supports the idea that students' self-regulatory processes can be enhanced and that better self-regulation results in higher academic performance. Beneficial effects on self-regulation have been obtained from interventions designed to improve students' goal orientations, learning strategies, self-monitoring and self-evaluations. (Schunk, p. 2005)*

He continues by noting the need for further research to examine the efficacy of promoting SRL in specific subject areas.

Research on academic self-regulated learning began as an outgrowth of psychological investigations into self-control among adults and its development in children (Zimmerman, 2001). This early initial self-regulation research was focused on its application in therapeutic contexts. For example, researchers taught participants to recognize and control dysfunctional behaviours such as aggression, addictions, and various behavioural problems (Zimmerman, 2001). In school settings, we can think of SRL as providing students with opportunities to: orient action towards a learning goal, monitor progress towards that goal, provide feedback (reflexivity), confirm or re-orient the trajectory of action towards the goal and/or re-define the learning goal (Allal, 2005). As with our discussion of cognition, in which we emphasised the importance of taking both the social and psychological dimensions into account, we adopt a similar approach here.

Two perspectives on self-regulated learning

We note two somewhat different strands of work on SRL, both of which it is important to consider in the promotion of SRL in schools: one adopting an individual cognitive psychological orientation, and the other adopting a multilevel socio-cultural orientation.

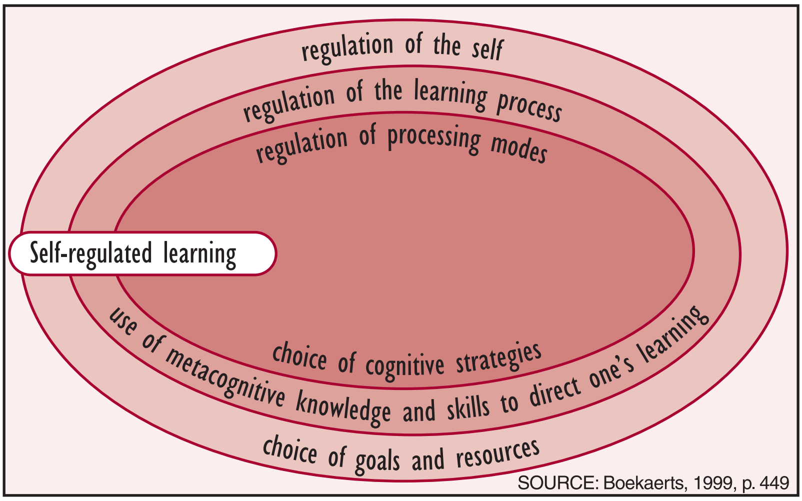
The first strand focuses on the individual learner (see figure 4, Boekarts, 1999). As illustrated by the model in figure 4, self-regulated learning comprises

- regulation of the self
- regulation of the learning process
- regulation of processing modes

- choice of cognitive strategies
- use of metacognitive knowledge and skills to direct one's learning
- choice of goals and resources.

The focus is very much on the individual learner, and reflects the psychological focus of much of the SRL literature. Allal (2005) presented a model of regulation (she writes about regulation rather than self-regulation) that stresses that more than the self is involved in regulation within school settings. Using a multi-level model, she argues that any effort to promote self-regulation must ensure that other aspects of the learners' environment must be congruent with such efforts.

Figure 4: A three-layered model of self-regulated learning



She identifies four levels as follows: overall teaching environment in the school; the teachers' approach to teaching; the quality of peer interaction; and students' own individual self-regulation strategies. Allal identifies various physical and psychological tools (e.g. teacher modelling strategies, multi-media learning environments, scripted peer feedback sheets, use of teacher think aloud to model mathematical problem-solving) that can be used as bridges between these levels.

3.8 Conclusion: rapid changes in approaches to learning

Each of the perspectives on mathematics teaching and learning discussed in this chapter could be portrayed in terms of the typical rhythms and patterns of teacher and student interaction. They each present an idealised image of learning. They also suggest that there is an emerging consensus in the learning sciences in relation to the design of powerful learning environments. Research is producing models that may have considerable import for the practice of teaching mathematics in schools. These new models of learning are more socially embedded than traditional ones. They include practical multi-level models of self-regulated learning and involve new understandings of the scope of assessment, including but extending beyond teacher assessment to include self and peer assessment. They suggest ways to integrate multi-media learning environments in mathematics education. How might some of these work in practice? What might future classroom learning for many or most students look like?

The Jasper Project is a well-researched initiative which has resulted in improvements in students' understanding of and skill in computation, enhanced problem-solving and improvements in transfer of learning to new situations (Roschelle, *et al*, 2000) (for a related learning and social interaction analysis of a technology-rich learning environment, see Järvelä, 1995). Early versions of Jasper used video to teach mathematics curricula focused on problem-solving, but researchers found that while students improved in their problem-solving skills and flexible use of knowledge the gains were not as great the designers had hoped. So the Jasper designers created a second wave of mathematics curricula. This second wave was called Jasper Challenge (the first wave was Jasper), and involved a multi-media learning environment (MMLE) called SMART that provided the following four types of support for learners:

- SMART Lab, providing comments on and summaries of student responses.
- Roving Reporter, involving videoclips of students grappling with the same problems in the learning community.
- Toolbox, comprising a toolbox for creating problem-related visual representations to aid mathematical representation and modelling.
- The Challenge: a new but related problem.

Jasper and Jasper Challenge provide a way to summarise key ideas in this chapter around four key themes: learner agency, reflection, collaboration and culture (Bruner, 1996; Brown, 1997). As we have noted, a move towards a more social and communal approach to learning is one of the defining characteristics of contemporary research on the design of powerful learning environments[9] (Roschelle, *et al*, 2000; De Corte *et al.*, 2003). The Jasper emphasis on Fostering a Community of Learners (Brown, 1997) is typical of efforts to understand and design *classroom cultures* where students share, negotiate and produce work that is presented to others. Learners' *reflection* on their own learning is a hallmark of effective learning and is seen in current efforts to promote self-regulated

9 Roschelle et al (2000), for example, identify several examples of computer-based applications to illustrate ways technology can enhance how children learn by supporting four fundamental characteristics of learning: (1) active engagement,(2) participation in groups, (3) frequent interaction and feedback, and (4) connections to real-world contexts.

learning. Jasper provides many opportunities for feedback in order to enhance students' SRL. The Roving Reporter and Toolbox in Jasper Challenge emphasise *collaboration* as a survival strategy; learners need each others' support and ideas to learn in the Jasper environment. This collaborative imperative draws attention to how knowledge is distributed in the classroom (Salomon, 1993). Finally, among Jasper learners, both teachers and students are seen as active meaning-making people with *agency*. As such, students are presented with real-world rather than artificial problems and supported in their efforts to create meaning.

In this chapter we have outlined very significant developments in the learning sciences and how these have influenced, sometimes in subtle ways (e.g. the behavioural underpinnings of much mathematics skill software), sometimes in very explicit ways (e.g. the impact of RME and situated cognition in PISA's mathematical literacy framework), conceptions of mathematics education. Increasingly there are calls for assessments to catch up with new perspectives on learning. The recent use of RME and situated cognition to frame PISA assessment items may be providing a context in which education systems, at least in some countries, especially those where new or modern mathematics is dominant, are beginning to grapple with whether and how they might begin to change their conceptions of mathematics curriculum and learning in order to measure up to new images of the competent mathematics student. Key issues raised in this chapter were as follows:

- New education and schooling goals demand new approaches to teaching and learning.

- Contemporary changes in mathematics education are being shaped by new visions of mathematics, mathematics learning and mathematics teaching.

- New visions of mathematics, mathematics learning and mathematics teaching have significant implications for assessment.

- A social turn has occurred in which the social and cultural dimension of learning and learners is becoming more central to policy and practice (e.g. the nature of PISA test items reflects a more socially embedded view of mathematics than most post-primary curricula internationally).

- Realistic Mathematics Education and situated cognition have emerged to play a key role in new understandings of the human mind which raise fundamental questions about the nature of mathematics education.

- Cognitive neuroscience has been the focus of increasing attention internationally (e.g. OECD, UNESCO) in debates about learning, although it will be a considerable while before this research translates into pedagogical policy and practice.

- Current research (e.g. NRC report on assessment and learning) is highlighting the more central role that learning ought to play in defining the nature of assessments.

- A variety of alternative assessments are being used in mathematics education including: portfolios, student journals, concept maps, assessment of cooperative group work, and revision of student work.

- Fostering students' responsibility for and ownership of learning is now a key education policy goal around the world. There is extensive research demonstrating that students of all ages, backgrounds and abilities can be taught self-regulation strategies.

A comprehensive approach to fostering ownership of learning is best viewed as a systemic issue involving school, classroom and student factors (Allal and Saada-Robert, 1992; Allal, and Pelgrims Ducrey, 2000; Allal, and Mottier Lopez, 2005; Allal, Mottier Lopez, Lehraus, and Forget, 2005).

CHAPTER 4

Five initiatives in mathematics education

Paul F. Conway, Finbarr C. Sloane, Anne Rath and Michael Delargey

4.1 Introduction

This chapter identifies five initiatives or important directions in mathematics education internationally and we discuss these using a common framework with the following headings: rationale, background, goals, key features, impact/outcomes, issues and implications for curriculum and assessment in an Irish post-primary context. The initiatives are:

- Mathematics in Context (MiC): RME-inspired curriculum materials
- 'Coaching' as a model for Continuing Professional Development in matehmatics education (West and Staub, 2004)[10]
- ICT (Information and Communication Technology) and mathematics education[11]
- Cognitively Guided Instruction (CGI)
- IEA First Teacher Education Study: The education of mathematics teachers (Schwille and Tatto, 2004; lead countries: USA and Australia).

A common thread through the five initiatives is their focus on the quality of student learning, one of the key areas in international policy discourse on education (UNESCO, 2004).

10 The case study on 'Coaching as model for CPD in mathematics education' was written by Dr. Anne Rath, Education Department, University College, Cork (UCC).

11 The case study on 'ICTs and mathematics education' was written by Mr. Michael Delargey, Education Department, University College, Cork (UCC).

4.2 Case 1: Mathematics in Context (MiC)

Rationale

The introduction to the MiC series describes its background as follows:

> *…Mathematics in Context is really a combination of three things: the NCTM Curriculum and Evaluation Standards, the research base on a problem-oriented approach to the teaching of mathematics, and the Dutch realistic mathematics education approach. (Education Development Center, 2001, p. 3)*

Mathematics in Context represents a comprehensive four-year mathematics curriculum for the middle grades (late primary and early post-primary) consistent with the content and pedagogy suggested by the NCTM *Curriculum and Evaluation Standards for School Mathematics and Professional Standards for Teaching Mathematics.* The development of the curricular units occurred between 1991 and 1998 through a collaboration between research and development teams at the Freudenthal Institute at the University of Utrecht, research teams at the University of Wisconsin, and a group of middle school teachers. MiC has been evaluated extensively and findings suggest that the curriculum is having a significant positive impact on student learning (see Romberg and Schafer, 2003).

The central tenet underlying the work is as follows. If mathematics is viewed as a language, then students must learn two interdependent knowledge sets: the design features of mathematics, and the social functions of mathematics. These include: the concepts and procedures of the language (the design features of mathematics) and solution of non-routine mathematics problems (a social function of mathematics). They must also develop the capacity to mathematise in a variety of situations (again, a social function). As such, the learning principles underpinning MiC are based on Realistic Mathematics Education and also reflect the principles of situated cognition, both of which, as we noted in chapter three, underpin PISA's mathematical literacy framework.

The MiC goals rest on an epistemological shift. The shift involves moving from assessing student learning in terms of *mastery* of concepts and procedures to making judgements about student *understanding* of the concepts and procedures and their ability to mathematise problem situations. In the past, too little instructional emphasis was placed on understanding, and the tests used to assess learning failed to provide adequate evidence about understanding or about a student's ability to solve non-routine problems.

Background

Because the philosophy underscoring the units is that of teaching mathematics for understanding, the curriculum has tangible benefits for both students and teachers (Romberg and Schafer, 2003). For students, mathematics is presented in opposition to the notion that it is a set of disjointed facts and rules. Students come to view mathematics as an interesting, powerful tool that enables them to better understand their world. All students should be able to reason mathematically; thus, each activity has multiple levels so that the able student can go into more depth while a student having trouble can still make sense out of the activity. For teachers, the reward of seeing students excited by mathematical inquiry, a redefined role as guide and facilitator of inquiry, and collaboration with other teachers, result in innovative approaches to instruction, increased enthusiasm for teaching, and a more positive image with students and society (note the commonalities with lesson study, Realistic Mathematics and Cognitively Guided Instruction).

Key features

A total of forty units have been developed for Grades 5 through 8. In terms of other available mathematics textbooks, the units are innovative in that they make extensive use of realistic contexts. From the activity of tiling a floor, for example, flows a wealth of mathematical applications, such as similarity, ratio and proportion, and scaling. Units emphasise the inter-relationships between mathematical domains, such as number, algebra, geometry and statistics. As the project title suggests, the purpose of each unit is to connect mathematical *content* both across mathematical domains and to the real world. Dutch researchers, responsible for initial drafts of the units, have twenty years of experience in the development of materials situated in the real world (see the commentary on RME in chapter three). These RME units were then modified by staff members at the University of Wisconsin in order to make them appropriate for US students and teachers.

Each of the units uses a theme that is based on a problem situation developed to capture student interest. The themes are the 'living contexts' from which negotiated meanings are developed and sense-making demonstrated. Over the course of the four year curriculum, students explore in depth the mathematical themes of number, common fractions, ratio, decimal fractions, integers, measurement, synthetic geometry, coordinate and transformation geometry, statistics, probability, algebra and patterns and functions. Although many units may focus on the principles within a particular mathematical domain, most involve ideas from several domains, emphasising the interconnectedness of mathematical ideas. The units are designed to be a set of materials that can be used flexibly by teachers, who tailor activities to fit the individual needs of their classes. One might think of the units as the questions (or questions sets) that a teacher can pose in the context of either lesson study or CGI. The design of textbooks is critical to MiC and reflects the RME philosophy:

> *In traditional mathematics curricula, the sequence of teaching often proceeds from a generalization to specific examples, and to applications in context. Mathematics in Context reverses this sequence; mathematics originates from real problems. (Education Development Center, 2001)*

Thus, the manner in which MiC 'reverses the sequence' in starting with real world problems as contexts for developing mathematical ideas, and possibly, but not necessarily, completing a unit with applications, draws attention to RME-inspired reframing of the role of real world contexts in mathematics education.

Over the course of the four-year MiC curriculum, students explore and connect the following mathematical strands:

- number (whole numbers, common fractions, ratio, decimal fractions, percents, and integers)
- algebra (creation of expressions, tables, graphs, and formulae from patterns and functions)
- geometry (measurement, spatial visualisation, synthetic geometry, and coordinate and transformational geometry)
- statistics and probability (data visualisation, chance, distribution and variability, and quantification of expectations).

Students work individually and in (flexible, not fixed) group situations, which include paired work and co-operative groups. The curriculum writers believe that the shared reality of doing mathematics in co-operation with others develops a richer set of experiences than students working in isolation. This focus on the valuable contribution that can be played by peers is consistent with the socio-cultural emphasis on the essential role of social support in fostering learning (see chapter three).

It is important to understand that the critical features are about the MiC curriculum itself (i.e., the content) and of components associated with the delivery process. We remind the reader that the delivery process (i.e., instruction) is not fully fleshed out in the original materials (or even in the detailed teacher's guides available through the commercial arm of *Encyclopedia Britannica*).

Impact and outcomes

A recent review in the USA of K-12 curricular evaluations indicated that there have been many evaluations of MiC as it has developed over the last decade (Mathematics and Science Board, 2004). These longitudinal and cross-sectional evaluation studies indicate that MiC improves students' achievement scores on standardised tests and their capacity to address non-routine mathematical problems (Romberg, 1997; Romberg and Schafer, 2003; Webb and Meyer, 2002). MiC writers assumed that when the curriculum is implemented well, students completing *Mathematics in Context* (MiC) will understand and be able to solve non-routine problems in nearly any mathematical situation they might encounter in their daily lives. In addition, they will have gained powerful heuristics, vis-á-vis the interconnectedness of mathematical ideas, that they can apply to most new problems that require multiple modes of representation, abstraction and communication. This knowledge base will serve as a springboard for students to continue in any endeavour they choose, whether it be further mathematical study in high school and college, technical training in some vocation, or the mere appreciation of mathematical patterns they encounter in their future lives.

While we agree with the general philosophy, the care and the quality of the materials, we also believe that without appropriate teacher development, similar to that outlined in either Cognitively Guided Instruction (see chapter four) or lesson study (see chapter two), the potential value of the MiC materials may be lost.

Issues and implications

Assuming that Irish post-primary textbooks are consistent with a 'new' mathematics approach, reviewing the impact of this approach on the sequencing of ideas and the underlying pedagogic vision seems essential in considering reform of post-primary mathematics education. Given textbooks' function as mediators between curricular intention and implementation, a reform of post-primary mathematics toward a more problem-solving orientation will, it could be argued, necessitate a radical overhaul of mathematics textbooks. As noted in the case of MiC, one very practical way in which textbooks have changed is the way in which the sequence of teaching unfolds. MiC textbooks proceed from real world problems to mathematical ideas rather than the traditional approach involving the generalisation of specific examples followed by real world applications.

4.3 Case 2: coaching as a case of subject-specific mentoring: a professional development model for teachers of mathematics

Rationale

The underlying principle of Content-Focused Coaching (CFC) for mathematics teachers is that authentic professional change in the teaching of mathematics involves teachers changing their fundamental knowledge base and beliefs about the teaching and learning of mathematics. This is coupled with the need for time and support to implement those beliefs in specific classroom contexts with a specific student body (Staub, 2004; West and Staub, 2003). It is assumed that practice change is difficult and requires ongoing coaching and support by an expert mentor well versed in the practice, content and principles of teaching mathematics. Another assumption is that change occurs over time and in bursts of uneven development, often requiring teachers to reframe their habitual teaching and learning strategies. Contexts indelibly shape how change occurs and specific contexts will demand different competencies, skills and strategies of teachers. Therefore, an on-site coach is available to help with adaptive teaching and learning strategies so as to better meet the needs of teachers and students in the learning of mathematics and to improve their conceptual understanding of mathematics.

The coaching context is structured around an expert mentor/coach working with one classroom teacher to collaboratively design, teach and reflect on teaching mathematics in a specific site. The coach provides a context for thoughtful and deliberate dialogues that result in improved teaching and learning - dialogues that reconnect a teacher to their own goals and passion for teaching, as well as connecting teachers to the body of empirical research that informs new ways of thinking about teaching and student learning in mathematics. A focus for these conversations is the content knowledge of mathematics, and through the use of a set of conceptual frameworks coming to a shared understanding of the fundamental concepts that underlie the discipline of mathematics, as well as a shared understanding of the principles of teaching and learning.

Background

Coaching is more familiar as a professional development model in business and sport than in an educational context. In business the role of the coach is to facilitate reflection and growth on the part of the client within a particular business context. Identifying specific problems and goals is the client's task. In sport the coach motivates, observes, models, gives ongoing specific feedback on progress and guides the development of the athlete as an individual or within a team.

In education, the efficacy of many professional development models has been questioned by educational reformers and researchers. They have challenged the dominant in-service 'one size fits all' model where new curricula and teaching methods are introduced to teachers in large workshop-style groups with little room for conversation, inquiry or on-site coaching (Fullan, 1995; Huberman, 1995). Research demonstrates that deep change rarely occurs in a teacher's thinking about teaching. Teachers are often treated as mere technicians and receivers of subject knowledge and pedagogical knowledge devised and generated by out-of-school 'experts' in contexts far removed from the busyness of classroom life (Clandinin and Connolly, 1996). Little attention has been paid to context-specific needs and how these contexts affect the implementation of new curricula and/or strategies. Nor has there been enough attention paid to the meanings that teachers themselves bring to the teaching/learning context; their motivation and passion for teaching has often been overlooked, thus failing to acknowledge the individuality of each teacher. Their skills and competencies in adapting their teaching to meet new curricula or standards have often been a subject of 'deprofessionalisation' (i.e. the sort of CPD that exclusively gives information to teachers rather than one that draws upon their professional knowledge as well as informing them) rather than professionalisation. Research demonstrates that efforts to introduce new teaching strategies are more successful if in-class coaching is part of the training (Joyce and Showers, 1995; Showers, Joyce and Bennett, 1987). This research shows that the on-site guidance and collaboration of a mentor who has expertise in the discipline content and the teaching of that content has been valuable.

It is within this context that CFC in the teaching of mathematics (West and Staub 2003) has been developed by a team at the University of Pittsburg's Institute for Learning. CFC is a professional development model designed to promote student learning and achievement by having a coach and a teacher working collaboratively in specific settings, guided by conceptual tools developed at the Institute (West and Staub, 2003, p. 2). These tools include a framework for lesson design and analysis and a set of core issues in mathematics lesson design that guide coach and teacher in deciding what to focus on in coaching conversations and reflection about mathematics thinking and teaching (see Guide to Core Issues in Mathematics Lesson Design below, adapted from West and Staub, 2003, p. 11)

- What are the goals and the overall plan of the lesson?
- What is the mathematics in this lesson?
- Where does this lesson fall in this unit and why?
- What are students' prior knowledge and difficulties?
- How does the lesson help students reach the goals?
- In what ways will students make their mathematical thinking and understanding public?
- What will students say or do that will demonstrated their learning?
- How will you ensure that students are talking with and listening to one another about important mathematics with mutual respect?
- How will you ensure that the ideas being grappled with will be highlighted and clarified?
- How do you plan to assist those students who you predict will have difficulties?
- What extensions or challenges will you provide for students who are ready for them?
- How much time do you predict will be needed for each part of the lesson?

The Institute was set up in 1995 as a partnership of school districts, committed to standards-based education and system-wide reform. Acknowledging the vast knowledge base that the past three decades has generated on teaching and learning, the Institute has been committed to translating this research into site-specific professional development tools for teachers. To this end it has also generated a set of nine principles of learning. These learning principles are condensed theoretical statements summarising decades of learning research (Resnick 1995a, 1995b; Resnick and Hall, 2001; see LRDC Learning Principles below):

- Organising for Effort
- Clear Expectations
- Fair and Credible Evaluations
- Recognition of Accomplishment
- Academic Rigour in a Thinking Curriculum
- Accountable Talk
- Socializing Intelligence
- Self-Management of Learning
- Learning as Apprenticeship.

The collaboration of Lucy West, a master instructional mathematics professional who is building a coaching system as a professional development model in a New York district of education, and Fritz Staub, a Swiss educator who came to the Institute as a postdoctoral fellow and was steeped in the tradition of didactics and deep subject-matter analysis, led to the evolution of content-focused coaching as a model for mathematics teaching. It is the bringing together of these two activities - the developmental processes of coaching, and a deep subject-matter focus and analysis - that offers such a comprehensive and promising model of staff development. This CFC model is also being adopted in other areas of the curriculum, including literacy and social studies education, by districts in the US who started by implementing it in mathematics.

Initiative's goals

Content-Focused Coaching provides structures for ongoing professional development that are underpinned by the conceptual tools above. According to West and Staub (2003, p. 3) coaching has the following goals:

- It helps teachers design and implement lessons from which students will learn.
- It is content specific. Teachers' plans, strategies and methods are discussed in terms of students learning a particular subject.
- It is based on a set of core issues of learning and teaching.
- It fosters professional habits of mind.
- It enriches and refines teachers' pedagogical content knowledge.
- It encourages teachers to communicate with each other about issue of teaching and learning in a focused and professional manner.

Key features

CFC takes place in schools. The teacher and coach are jointly accountable for initiating and assisting effective student learning. This feature ensures that the coach is intimately involved in all aspects of the lesson, including design, teaching and evaluation. The coach is a master teacher of mathematics and is well versed in the curriculum, standards, and principles of mathematics teaching and learning. The coach's responsibility is to set up and sustain a good working relationship between coach and teacher. This relationship is based on mutual respect of both roles. Key features that characterise this relationship include content-rich conversations, coaching conversations, site-specific interventions and observations, meticulous co-planning using a series of conceptual tools to guide design, and a commitment to work over a period of time to improve student learning.

The following structures are integral to this approach: the coach and teacher have a pre-lesson conference; they observe, teach, or co-teach the lesson; and they have a post-lesson conference. The coach comes to the teacher's classroom and together they talk and think about the mathematics to be focused on.

The pre-lesson conference

The first task of the coach is to get the teacher talking about how they view and feel about their teaching of mathematics and to gain their trust in having a coach as a mentor. This talk is focused on assessing the prior knowledge and experience of the teacher and the anticipated difficulties of the students. It is very important for the coach to ascertain how motivated and confident the teacher is in his/her own teaching and to understand the specific context that the teacher is working in. Then the coach and teacher collaboratively set goals and criteria for reaching these goals using the conceptual tools identified above. Establishing clear, explicit learning goals for students linked directly with mathematical content increases the possibility that important mathematical concepts will be included. Developing a shared view on the strategies, tasks, concepts and mathematical skills that the students are working towards, and also on a lesson design, paves the way for the focus on student learning which will be the subject of the post-lesson conference.

The lesson

The teacher and coach will have decided who will teach, observe and co-teach this lesson. Ordinarily the coach will have invited the teacher to choose a path that they are comfortable with and the teacher will have decided what they would like feedback on. However, the coach's role will vary a lot depending on the context and the teacher's needs. It might include modelling a specific strategy, co-teaching with the teacher, observing a class and focusing on a particular area that the teacher needs feedback on and so on. At all times, however, the coach's role is a collaborative one with the teacher. The goals and lesson design have been collaboratively reached and both teacher and coach are jointly responsible for student learning, which is where the focus lies.

Post-lesson conference

The post-lesson conference focuses on how successfully the lesson plan was implemented. Did students learn what they were supposed to? Examining student work is often a part of this conference - looking for evidence of learning. What problems arose? What strategies were successful and what strategies were less so? Were the learning goals appropriate to this group? How successfully were the mathematical concepts taught and linkages made to previous material? This conversation often leads into a pre-lesson conference for the next lesson where new goals are set. The coach supports and guides the teacher in thinking through these questions and also offers strategies, specific feedback and challenges the teacher when appropriate.

Impact and outcomes

One of the most important potential outcomes of CFC is the development of a learning community committed to both improving classroom practice (as in lesson study) and meeting high standards of teaching and learning. Talking becomes a main conduit for teachers to learn how to sustain and promote further learning. However, not all talk sustains learning. Teachers and coach become adept at developing a kind of talk that is called 'accountable talk'; that is, accountable to deep learning for all students, to accurate and appropriate knowledge in the domain of mathematics, and to rigorous thinking. Because both coach and teacher are working jointly together with a group of students they have a common context within which to think through problems of teaching. The talking that they do is accountable to developing a thinking curriculum together and is linked to a further research community and a shared understanding of the discipline of mathematics. Thinking and problem solving on the part of teachers and students is focused on, and both learn to become responsible for promoting and deepening further learning. Teachers become empowered and become adept at using powerful conceptual tools that further their learning: these tools can be used in their classrooms and with peers in thinking about problems of practice. Teachers become critical reflective practitioners in their field and become confident in adapting and developing pedagogical content knowledge (Shulman, 1987). More importantly, this model of professional development holds out the possibility of bridging the theory-to-practice gap that has characterised much of the history of teaching and learning (Schön, 1987).

Issues and implications

One of the most important issues that this initiative underlines is the need for ongoing support and professional development of mathematics teachers if students are to reach the necessary performance levels for a knowledge-based 21st century society. In Ireland, the present system that certifies teachers to teach mathematics with a degree in mathematics and a one-year Postgraduate Diploma in Education[12] (PDE) is insufficient. Teachers need to understand how to help students to think mathematically and to understand deeply the development process of learning mathematics. For teachers to teach in this way they need professional development opportunities so that they can engage in the very practices that a new teaching for thinking curriculum requires. They need to have space to think through the design, teaching and evaluation of their teaching with colleagues and expert mentors/coaches on a regular basis. Onsite work provides a model for giving teachers the help they need in very specific and concrete ways, and allows the coach to give feedback that is situation-specific and also content-specific. By the same token, there needs to be greater clarity as to the content of mathematics and the developmental process that students go through in understanding mathematical concepts. Students need to be given opportunities to generate their own knowledge, whilst at the same time they are in a context where the teacher is skilfully guiding and probing for ever more depth in understanding. The imperative of covering the curriculum needs to be replaced by a push to develop learning environments where students are set tasks that develop their problem-solving skills and self-regulating skills. In the same way teachers need such learning contexts for their ongoing development.

12 The Higher Diploma in Education (HDE), often just referred to as the 'HDip' or 'The Dip', was renamed the Postgraduate Diploma in Education (PDE) in 2005 in order to align itself with the National Qualifications Authority qualifications framework.

4.4 Case 3: teaching mathematics using ICTs

Rationale

Why use ICT to teach mathematics? One reason offered by the Joint Mathematical Council of the United Kingdom's review of algebra teaching, which was carried out for the Royal Society, is that:

> *The growth of IT has made it possible for students to manipulate many different types of external representations on the screen, involving symbolic, graphical and tabular forms. It is now possible to manipulate graphical representations in ways which were not possible on paper. Harnessing this new power within mathematics and school mathematics is the challenge for the 21st century.*
>
> **(Royal Society, 1995 cited in Oldknow and Taylor, 2000 pp. 90-91)**

According to Cockcroft (1982) once technology in general enters the school mathematics classroom and curriculum there are two ways in which technology such as computers and calculators can impact upon the school mathematics curriculum. Firstly they can assist and improve mathematics teaching. Secondly they can shape the mathematics curriculum itself. Tobin (1998) is of the opinion that new technological tools provide an impetus and opportunity for curriculum reform and teachers need to determine the most effective use of this new technology in the classroom. Technology helps students to learn mathematics and develops their knowledge of the subject, to such an extent that the US National Council of Teachers of Mathematics view it as an important component of a high-quality mathematics education programme: 'technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.' (NCTM, 2000, p. 24). Teachers need to apply ICT in ways that enhance the teaching and learning of the current established curriculum (NCTM, 2000; Oldknow and Taylor 2000) so that ICT can enable students to concentrate on more interesting and important aspects of content (Oldknow and Taylor, 2000) and that ICT can make the teaching of mathematics more efficient and effective (Butler, 2005).

Background

Historically, mathematics teachers were typically at the forefront of integrating computers into their teaching (Kelman, Bardige, Choate, Hanify, Richards, Roberts, Walters, Tornrose, 1983). Reflective of this early adoption of technology by mathematics teachers in some settings, Venezky and Davis (2002) found that ICT diffusion into a school curriculum in Luxembourg was due to innovation by a small group of male science and mathematics teachers motivated by an interest in the 1980s era of computer programming. Today, teachers' interest in technology is rarely the result of an interest in programming and early adopters of ICTs in teaching come from a variety of subject backgrounds.

An examination of ICT's place in international contemporary mathematics curricula provides some insight into current ICT use in school mathematics. One way of bringing technology into school mathematics practice is to make it a mandatory part of the syllabus. This has been the case in a number of countries such as Austria, Denmark, Luxembourg, Singapore, and New Zealand. According to the Danish senior cycle syllabus documents:

> *IT must be included as part of the instruction, e.g. by using application programs or programs for illustrating and teaching subject-specific concepts or methods. In addition, the instruction must include examples of how certain mathematical procedures can be expressed in algorithmic terms.*
>
> **(Danish Ministry of Education, 1999)**

According to Venezky and Davis (2002), ICT use in teaching and learning will be explicitly mentioned in the mandatory curriculum and ICT skills in mathematics will be part of the final examination for the school leaving certificate for post-primary schools in Luxembourg. The use of computers has been fixed in mathematical instruction in Austria since the last curricular reforms in 1993 and 2000 (Wurnig 2003). Leaving Europe and turning towards Asia, the focus of Singapore's revised syllabus is mathematical problem-solving. The integration of IT into mathematics teaching and learning will give leverage to the development of mathematical problem-solving (Singapore Ministry of Education, 2001 p. 8).

In Korea, the current 7th National curriculum, published in 1997, does not presuppose computer use (Hwang, 2004), although teachers can use them if they wish to do so (Hee-Chan Lew, 1999). Lew also writes that software development in Korea is moving from concentrating on low-level computational skills to higher-order software, such as games, tool-like applications, simulation and databases. Thus, Korea seems to be migrating from Sinclair and Jackiw's (2005) wave 1 type to wave 2 type (see below), which considers the context of learning. Hwang's comparative analysis of mathematics curricula in Korea and England (2004) notes general recommendation in the instructional guidelines section of the Korean curriculum that the 'active utilization of calculators and computer is recommended in order to improve the understanding of concepts, principles, and rules, and to enhance problem-solving abilities' (p. 9). He is impressed with the concrete presentation of content area for computer use in the English curriculum and he gives a few examples (Hwang 2004 p. 9):

- use systematic trial and improvement methods with ICT tools to find approximate solutions of equations where there is no simple analytical method, for example $x^3 + x = 100$ (the sub-area of 'numerical methods' at key-stage 3)

- plot graphs of: simple cubic functions… the reciprocal function… the exponential function… the circular function… using a spreadsheet or graph plotter as well as pencil and paper; recognise the characteristic shapes of all these functions (the subarea of 'other functions' at key-stage 4 higher).

The 1992 mathematics curriculum introduced into New Zealand schools assumes in the technology curriculum statement that both calculators and computers will be available and used in the teaching and learning of mathematics at all levels (New Zealand Ministry of Education, 1992). The curriculum statement views graphics calculators and computer software as 'tools which enable students to concentrate on mathematical ideas rather than on routine manipulation, which often intrudes on the real point of particular learning situations' (p. 14). Computer programs such as Logo facilitate mathematical experimentation and open-ended problem-solving. The curriculum document makes reference where appropriate to ideas concerning the students' use of technology, for example: 'students use a calculator or graphics package and the remainder theorem to identify factors or to locate intervals containing the roots to an equation' (p. 167).

Key features

There are four roles technology can play in the development of mathematical knowledge (Alagic, 2002). Firstly, technology empowers teachers and students to deal with multiple representations. So for example, Dugdale (2001) uses spreadsheets to investigate chaotic behaviour. She states that the students' investigation combined tabular and graphical representations. Understanding the relationships among several of these representations was essential to understanding the chaotic behaviour of the functions studied.

Secondly, technology enhances our ability to visualise. Dynamic geometry software packages such as *The Geometer's Sketchpad* (GSP) and *Cabri Geometry*, as well as graphing packages such as *Autograph or TI InterActive*, have made the teaching and learning of various geometrical and algebraic topics more interesting. For example, research carried out by Dixon (1997) concluded that 8th grade students in the United States who were taught about the concepts of reflection and rotation in a GSP environment significantly outperformed their traditionally taught peers on content measures of these concepts. Similarly they scored well on measures of 2-D visualisation.

Thirdly, technology increases the opportunities for development of conceptual understanding. Butler (2005) refers to the development of web-based *Java* and *Flash* applets and their transformation of the teaching of calculus. The concept underlying the very important chain rule in differentiation or that underlying integration (area under a curve) may be understood visually, thus reducing the extent of routine drill and practice. Almeqdadi's 2005 study concerning the effect of using GSP on Jordanian students' understanding of some geometric concepts concluded that there was a significant effect of using this software and consequently he recommends that more emphasis should be placed on computer use in mathematics and in education.

Finally, the opportunity for individualised learning is enhanced by technology. Sinclair and Jackiw (2005) identify three waves of ICT development, namely ICT for learners of mathematics, developing the context of learning and tomorrow's ICT. The relationship between the individual learner and the mathematics itself is the heart of first-wave technologies. They class *Logo* and the multiple-choice tests of the 1970s computer-aided instruction (CAI) as belonging to this technological wave. They have both achieved individual learning experiences at the cost of 'neglecting classroom practice, teacher habits and beliefs, as well as the influence of the curriculum, by imposing entirely new and perhaps inappropriate classroom practices' (Sinclair and Jackiw, 2005, p. 238). Individualised learning and independent student learning may also be facilitated by on-line technologies. Nicholas and Robertson (2002) give a brief account of the SCHOLAR[13] project developed at Heriot-Watt University in Scotland. The programme was used in a number of Scottish schools and as well as being used by teachers it is also designed for flexible learning by the independent student. Sinclair (2005) in her account of mathematics on the internet suggests that sites such as Ask Dr. Math[14] provide opportunities for curious students to investigate non-school-related mathematics independent of the school situation.

On the assessment side, ICT can be used in a number of ways. Firstly, it may be used as a mandatory component of the assessment process. Oldknow (2005) states that in the UK there is now compulsory data-handling coursework in mathematics which specifies the use of ICT. Singapore considers that the main purpose of mathematical assessment should be to improve the teaching and learning of mathematics and recommends that information technology be incorporated where appropriate (Singapore Ministry

13 SCHOLAR may be accessed at http://scholar.hw.ac.uk/heriotwatt/scholarlogin.asp

14 http://mathforum.org/dr.math/

of Education, 2001 p. 8). ICT skills in mathematics will be part of the final examination for the school-leaving certificate for post-primary schools in Luxembourg (Venezky and Davis, 2002). One of the assessment criteria for the internal assessment assignments in the International Baccalaureate is the use of technology (Brown 2002). Parramore (2001) informs us that the assessment of modelling is difficult to do under the conditions of a written examination. Coursework is the medium to do this. He proposes one approach at incorporating computer-based examining into A-level mathematics. He suggests that if one uses a computer as part of the modelling process then it should be reflected in ongoing school-based assessment. Computers may also be used to perform assessment tasks. Sangwin (2003) reports a recent development in mathematical computer-aided assessment (CAA) which employs computer algebra to evaluate students' work using the internet. He claims that these developments are of interest to all who teach mathematics, including class-based school teachers. He shows how technology may be used as a tool for uncovering students' misconceptions about simple topics. Uses of internet-driven technology may include the marking of existing problem sets and providing instant tailored feedback based on properties of a student's answer. The SCHOLAR project in Scotland reports positive and encouraging student and teacher experiences.

Impact and outcomes

Oldknow (2005) gives an account of an innovative project in the application of ICT in the mathematics classroom in 2000-01. The 'MathsAlive' project focussed on the use of ICT in twenty Year 7 (11-12 year-olds) mathematics classes in England. This project developed the hardware (PCs and interactive whiteboards) and software (*The Geometer's Sketchpad, TI InterActive!*) infrastructure necessary to integrate ICT into mathematics classrooms. Oldknow (2005) includes extracts from the evaluator's final report of September 2001, including overall indicators of success of this pilot project (pp. 182-83):

- teachers have been extremely positive about the values and usefulness of the resources throughout the project, and have wanted to continue with the project beyond the period of the pilot
- teachers have felt that the resources and the training offered have enabled them to implement the objectives and needs of the National Numeracy Strategy, using technology to support their teaching
- teachers have felt that the technology has added to their teaching strategies and approaches
- students have reported positively throughout the period of the project on the value of the resources and the impact it has had on their learning and on their positive attitudes towards mathematics
- the resources have been shown in practice to support both teaching and learning.

There were, however, some weaknesses in the project. There were too many new things for the teachers to take in, and a lack of time for training, discussion, revision and maturation. Some teachers found it hard to share control of the whiteboard with the students. Relatively few adapted the prepared teaching materials and even fewer designed their own. But, as Oldknow (2005) points out, the project was a success and the ICT did impact on both the students' and teachers' enjoyment and understanding of the underlying mathematics. Other authors, such as Passey (2001), Glover, Miller and Averis (2003) and Clark-Jeavons (2005), have written on the use of interactive whiteboards in teaching mathematics. Letting the user 'touch the mathematics' is how Passey (2001) describes the use of this technology. Interactive whiteboards offer learners the opportunity to get closer to mathematical systems and processes in an exploratory way (Clark-Jeavons 2005). In general, however, the integration of ICT into mathematics teaching in the United Kingdom has been poor, as evidenced by recent HMI Inspection Reports. One such Ofsted (2002a) report made the following comments on ICT practice in mathematics teaching within UK schools:

> *The use of ICT to support learning in mathematics is good in only one quarter of schools. It is unsatisfactory in three schools in ten. Typically there is some use of ICT with some classes, but it is not consistent across the department. Students' access to a range of mathematics software also varies greatly. Most departments have access to spreadsheets, graph-plotting software, LOGO and specific items of software to support skills learning. In general, however, very little use is currently made of the powerful dynamic geometry or algebra software available. Many mathematics teachers use ICT confidently outside the classroom in the preparation of teaching materials and in the management and analysis of students' achievement. Despite this, only a small proportion of departments have reached the point where they can evaluate critically their use of ICT and decide where it most benefits learning in mathematics. Too often, teachers' planning and schemes of work lack any reference to specific ICT applications, and students have difficult recalling when they have used ICT in mathematics. (pp. 8-9)*

A second report concerning the implementation of government ICT initiatives relating to post-primary mathematics published in 2002 states that overall good practice in relation to ICT use in mathematics teaching remains uncommon and that although some use is made of ICT in around two-thirds of mathematics departments, it is not an established part of the curriculum (Ofsted, 2002b). It seems that little has changed since Taverner (1996) wrote that the use of ICTs in mathematics teaching in schools is claimed to be as little as once per term on average.

In relation to using ICT in the assessment of mathematics, Ashton, Schofield and Woodger (2003), in their paper on piloting summative web assessment in post-primary education, state that the challenge for on-line assessment is not a technical but a pedagogical one: does on-line assessment measure the same learning outcomes as traditional paper based one? Beevers, Fiddes, McGuire and Youngson (1999) remind us that students consider it important that the computer issues feedback if it is to grade their work fairly and according to Bower (2005) providing students with their nonpreferred form of feedback has a significantly negative impact on their mathematics ability self-rating.

Issues and implications

ICTs have impacted upon school mathematics in a number of ways, from enhancing the teaching of the subject, to changing the curriculum content, to being used as an assessment tool. The role which ICT has assumed varies across the international arena. The current situation in Irish mathematics teaching is far from encouraging. Lyons, Lynch, Close, Sheerin, and Boland (2003) state that the use of educational technology in mathematics classrooms remains the exception rather than the norm.

Mulkeen (2004) confirms this by reporting that just 17% of post-primary schools used ICT in mathematics monthly or more in the year 2002. This same survey found that 67% of schools used it occasionally. The Impact of Schools IT2000 report (National Policy Advisory and Development Committee [NPADC], 2001) found that post-primary principals stated that mathematics software was been used in their schools, but the report did not elaborate on the type of software and manner in which it was being used by mathematics teachers. The NPADC report indicated that technology teachers were the most likely to be integrating ICTs into their teaching. Mariotti (2002) reminds us that computers have slowly entered schools and have been integrated into practice even more slowly. A radical change of objectives and activities is required if computer technologies are to be integrated into school practice. This review has referred to countries which have made ICT compulsory in both the teaching and assessing of mathematics, such as Denmark, New Zealand, Austria and the UK. Perhaps Ireland should follow suit and incorporate more explicit use of ICT in senior and junior cycle mathematics. The ways in which this may be achieved include:

- developing the ICT mathematics infrastructure in schools: e.g. provision of more technical support for schools in the ICT area
- incorporating an ICT coursework assignment as part of the assessment of junior and senior cycle mathematics similar to those in Luxembourg, Austria and the UK
- including explicit ICT curriculum statements in the preamble to curricular documents like those in Denmark, Singapore and New Zealand

- increasing teacher awareness of the potential of ICT in teaching and assessing mathematics through more training, as it happened in the SCHOLAR project in Scotland and the MathsAlive project in England.

However, the challenge for integrating ICT more fully into the Irish mathematics curriculum is to learn from international best practice and localise it to suit the Irish context.

4.5 Case 4: Cognitively Guided Instruction (CGI)

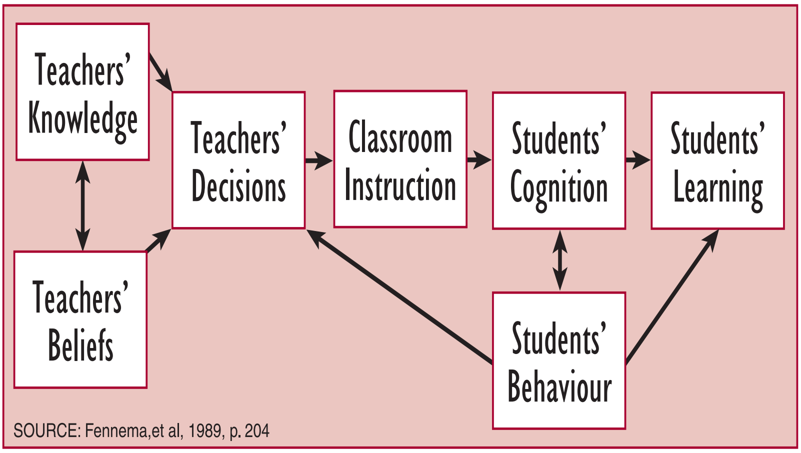
Rationale

The basic philosophy underlying Cognitively Guided Instruction (CGI) is that teachers need to make instructional decisions based on knowledge drawn from cognitive science about how students learn particular content. The critical viewpoint is that student understanding (or learning) involves linking new knowledge to existing knowledge (Vygotsky, 1978; Collins, Greeno and Resnick, 1996). Fennema, Carpenter and Petersen (1989) explain that teachers need to be cognisant of what knowledge their students have at various stages of the instructional process so they can provide appropriate instruction. Here we see a parallel with lesson study (LS) as Japanese teachers explicitly ask, in advance of teaching, just how they think students will respond to the mathematical content being taught. As we have seen, Japanese teachers carefully craft their strategies in anticipation of the most likely student responses.

Background

The CGI philosophy described above has, over the past twenty years, served as a guideline for research in which the 'major focus has been to study the effects of programs designed to teach teachers about learners' thinking and how to use that information to design and implement instruction' (Carpenter and Fennema, 1988, p. 11). The model illustrated below provides the framework for CGI research.

Figure 5: The Cognitively Guided Instruction research model



SOURCE: Fennema,et al, 1989, p. 204

The diagram illustrates that classroom instruction is the result of teacher decision-making in real time. These decisions are assumed to derive from a number of sources: teacher knowledge and beliefs, and teacher assessment of student knowledge. The latter are generated when teachers carefully observe (and interpret) their students' mathematical behaviours and the mathematics artefacts that result from those behaviours.

Key features

Fennema *et al*. argue that the main features or tenets of CGI are: '(1) instruction must be based on what each learner knows, (2) instruction should take into consideration how children's mathematical ideas develop naturally, and (3) children must be mentally active as they learn mathematics' (1989, p. 203).

The Wisconsin group (Fennena, Carpenter, Petersen, and their graduate students, including Franke [née Loef] and Chiang) have continued this line of CGI mathematics education research work over the past twenty years. Here we provide a short summary of an early study and note that more recent vignettes for teachers are available in the following publications: Jacobs, Carpenter, Franke, Levi and Battey, D. (under review); Franke, Kazemi and Battey (in preparation); Franke, Carpenter and Battey (in press); Franke, Kazemi, Shih, Biagetti and Battey (2005); Battey, Franke and Priselac, (2004); Carpenter, Franke and Levi (2003); Franke, Carpenter, Levi and Fennema (2001); Franke and Kazemi (2001); Carpenter, Fennema, Franke, Levi and Empson (1999).

Impact and outcomes: insight from the 1989 study

Carpenter, Fennema, Petersen, Chiang and Loef (1989) conducted a study to explore the three central components of CGI listed above. Their study involved 40 1st grade teachers (half of whom were randomly assigned to the treatment group). The treatment group attended a one-month summer workshop where participants were introduced to the research literature on the learning of addition and subtraction concepts. All forty teachers and their students were then observed throughout the autumn, winter and spring terms as they taught. Each teacher was observed on at least 16 separate occasions between November and April. The researchers collected data including preand post-test measures of student achievement (with various measures of mathematical problem-solving). They also collected measures of student beliefs and confidence, and conducted student interviews. Results showed that students in the treatment classes performed significantly better than control students on both the recall of number facts and the measures of problem-solving. The researchers showed that experimental teachers spent more time on word problems than the control teachers, who in turn spent more time on number fact problems. Experimental teachers focused more of their instructional attention on the processes that students used to solve problems, while control teachers attended to the answers that their students produced. Finally, experimental teachers allowed their students to respond with a wider range of strategies in order to solve mathematical problems (again, a central tenet of LS is to allow students the opportunity to generate and evaluate their own solutions to mathematical problems).

The following vignette excerpted from Carpenter, Fennema, Petersen, Chiang and Loef, (1989), provides parallel insights into the actions of the treatment teachers and their students.

> *A typical activity that was observed in CGI classes was for a teacher to pose a problem to a group of students. After providing some time for the students to solve the problem, the teacher would ask one student to describe how he or she solved the problem. The emphasis was on the process for solving the problem, rather than on the answer. After the student explained his or her problem-solving process, the teacher would ask whether anyone else solved the problem in a different way and give another student a chance to explain the new solution. The teacher would continue calling on students until no student would report a way of solving the problem that had not already been described… In contrast to the CGI teachers, control teachers less often (a) posed problems, (b) listened to students' strategies, and (c) encouraged the use of multiple strategies to solve problems. They spent more time reviewing material covered previously, such as drilling on number facts, and more time giving feedback to students' answers. (p. 528)*

The parallels between CGI and the problem-posing approach to teaching mathematics in Japanese classrooms are striking.

Issues and implications

CGI presents important insights for all levels of teacher education: pre-service, induction, early career, and continuing professional development. As we noted earlier, teachers' beliefs play a critically important role in practice and can act as a support for or impediment to reform in any subject area. In terms of pre-service education, CGI presents real challenges by pressing the case for a review of how much of an impact current teacher education is actually having on neophyte teachers' beliefs. From a CGI perspective, one could make a case for teaching practice tutors and mentors of beginning teachers to direct considerable energy into exploring beginning teachers' beliefs about learning and mathematics as an important feature both of the content of teacher education and a component in the appraisal of teacher competence.

4.6 Case 5: first IEA teacher education study: the education of mathematics teachers

Rationale

The Teacher Education Study in Mathematics, a cross-national study of elementary and secondary teacher preparation (TEDS-M 2008), is 'designed to inform and improve the policy and practice of how future teachers learn to teach a challenging mathematics curriculum in elementary and secondary school' (TEDS-M, 2005). As of 2005, the study had been under development for three years and was discussed and approved at successive IEA General Assembly meetings. The study is designed to help participating countries respond to a number of urgent concerns about mathematics education and will:

(a) help to strengthen the knowledge base to address participating countries' national priorities such as increasing the number of fully competent mathematics teachers; (b) gather empirical data on the experience of the participating countries to help resolve conflicts over the nature, benefits and costs of teacher education, in order to support improved policies for selection, preparation, induction, and professional development of mathematics teachers; (c) foster a more systematic and scientific approach to the study of teacher education and teacher learning in mathematics; (d) develop concepts, measurement strategies, indicators and instrumentation to strengthen the research in this field, and the knowledge base of teacher education cost-effectiveness (TEDS-M, 2005).

This study addresses the following questions:

- What is the level and depth of the mathematics and related teaching knowledge attained by prospective primary and lower secondary teachers that enables them to teach the kind of demanding mathematics curricula currently found across countries? How does this knowledge vary from country to country?

- What learning opportunities available to prospective primary and lower secondary mathematics teachers allow them to attain such knowledge? How are these structured? What is the content taught in teacher education programmes, and how is instruction organised?

- What are the intended and implemented policies that support primary and lower secondary teachers' achieved level and depth of mathematics and related teaching knowledge? How do teacher policies influence the structure of primary and lower secondary mathematics teachers' opportunities to learn mathematics at national and institutional levels?

Background

The IEA (International Association for the Evaluation of Educational Achievement) has undertaken major cross-national surveys of educational achievement since the early 1960s. These studies were designed to provide a rich empirical base from which both to understand factors influencing educational outcomes and inform educational policy makers. One of the original attractions of such cross-national studies was that cross-national variation found in the relationships between variables would help highlight the importance of particular variables and also draw attention to educational and wider cultural factors shaping relationships between relevant variables. In the case of IEA mathematics studies, Ireland participated in the curriculum analysis component of the Second International Mathematics Study (SIMS, 1980-82), and the achievement and textbook study in the Third International Mathematics and Science Study (TIMSS, 1995) (Cosgrove *et al.*, 2005). Ireland did not participate in the First International Mathematics Study (FIMS), the Third International Mathematics and Science Study (TIMSS-R, 1999) or in Trends in Mathematics and Science Study (TIMSS, 2003).

TEDS-M is the result of an increasing awareness of the importance of teacher education in both understanding existing classroom practice and changing classroom practice. Given the focus on mathematics education as a policy priority in many countries it is not surprising that the first IEA teacher education study chose to focus on mathematics (see chapter one). TEDS-M is being funded in

2006 (commencing September), 2007 and 2008 by the US National Science Foundation (NSF), as well as through a fee levied on participating countries. The development and piloting of data-gathering instruments/protocols have been undertaken through preliminary sub-study P-TEDS. Data-gathering protocols developed include survey instruments for mathematics lecturers in university departments, mathematics educators, future mathematics teachers, and two institution-focused surveys. The institution-focused surveys will gather data on the routes into teaching taken by teachers of mathematics (recognising that not all teachers of mathematics are trained mathematics teachers or have a degree in the area) and a survey of the teacher education institutions involved in the education of lower secondary mathematics teachers.

Issues and implications

The Report of the Review Group on Post-primary Teacher Education in Ireland (2002) observed that there is scope for more research on teacher education in Ireland. While there is a considerable amount of research by individuals or small-scale collaborative initiatives (e.g. the Standing Conference on Teacher Education North and South, SCoTENS), large scale programmatic cross-institutional studies of teacher education, with or without the international comparative dimension envisaged in TEDS-M, have not been undertaken to date. This is in our view problematic, both for teacher education in general and more specifically for implementing any reforms in post-primary mathematics education.

4.7 Conclusion

In this chapter, we have outlined a number of diverse initiatives of innovation in mathematics education (see Table 6).

Table 6: Five initiatives in mathematics education

Initiative (agency)	Rationale and goals	Issues/questions for maths education in Ireland
Mathematics in Context (MiC) (NSF)	Context-focused maths is neglected Goal: To provide RME-type curriculum materials for students in USA	How do Irish post-primary textbooks compare with MiC textbooks?
Coaching (LRDC)	Enhancing teacher knowledge and classroom practice Goal: To create mentoring pairs in the context of maths	How can evolving models of continuing professional development (e.g. induction) integrate a subject coaching perspective?
ICTs	New ICTs are changing conceptions of knowledge Goal: To transform learning using ICTs	Given the low usage of ICTs by Irish post-primary mathematics teachers, under what conditions might teachers use the increasing array of mathematics-related ICTs (including new handheld technologies)?
Cognitively Guided Instruction (UWM)	The lessons of cognitive science and their implications for teaching mathematics Goal: To develop teachers' capacity to use contructivist compatible teaching strategies	How can continuing professional development initiatives in mathematics provide opportunities for teachers to understand and reframe, where appropriate, their beliefs about teaching and learning mathematics?
IEA-Teacher Education Study (IEA, NSF, ACER)	Understanding links between teacher education, teacher knowledge and student learning Goal: To develop models of teacher education in order to enhance classroom maths teaching	A hypothetical question - What could be learned about the education of mathematics teachers if Ireland participated in the IEA's first Teacher Education study?

The Mathematics in Context (MiC) initiative provides a useful model of what a move towards more realistic mathematics education entails for the development of textbooks and supporting web or other ICT-based materials. The coaching initiative provides a model of the kind of subject-specific mentoring support that we think is essential in fostering a new era of co-operation and sharing of pedagogical practice. Coaching is a good example of what Hargreaves (2000) has identified as a move toward a new type of professionalism in teaching, defined by collegiality rather than the traditional image of the autonomous professional teacher. The coaching of early career and more seasoned teachers by thoughtful subject specialists extends notions of mentoring beyond a focus on general teaching skills characteristic of teacher induction, and highlights the important role of subject matter knowledge. Indeed, one of the strengths of the coaching case we described is its emphasis on the importance of mentor teachers' deep knowledge of subject matter and how it can be represented to enhance student learning. This is an important reminder of the value of the knowledge base residing among practising mathematics teachers, a vital feature and essential resource in any attempted reform of mathematics education.

The various examples of how ICTs can promote transformation in mathematics are a timely reminder of how access to advanced technologies which support representation of ideas in new ways is changing the work of mathematicians (Lei, Conway and Zhao in press; Roschelle et al, 2001). It presents a real challenge and opportunity for revising mathematics syllabi. In school settings, Cuban (2000) has characterised the current wave of technological innovation as one of overselling and underuse. Based on detailed analysis of ICT use in three classroom settings (kindergarten, high school and the medicine faculty in Stanford University) in Silicon

Valley (probably the world's most technology-rich area), Cuban presented convincing evidence that innovative ICTs are used more by teachers for preparation than for day-to-day teaching in the classroom. In the Irish context, a helpful start in increasing the use if ICTs can be made by ensuring they have a more central role in syllabus documents and related policies (e.g. NCCA documents identify ICT as a compulsory component of the new art syllabus and the four new technology syllabi awaiting implementation). Cognitively Guided Instruction (CGI) is an important case because it presents some of the challenges in moving toward more constructivist teaching. As we noted earlier, teachers' everyday or folk theories of learning have a profound impact on the structure and flow of teachers' daily lessons. Finally, the innovative IEA Teacher Education Study in Mathematics (TEDS-M 2008) is the first of its kind and is a vital opportunity for understanding the dynamics of the development of mathematics teachers' pedagogical and subject knowledge-base and how it impacts both their practice and student learning.

Chapter 5

Redefining and reforming mathematics education: issues and implications for Irish post-primary education

5.1 Introduction

> *Education systems tied to the formation of nation-state citizens and consumers bonded to local systems to the neglect of larger global forces are likely to become obsolete, while those that proactively engage globalization's new challenges are more likely to thrive.*
>
> **Suarez-Orozco, 2004**

> *There is little doubt that Kevin Myers's recent, thought-provoking article in the Irish Times, in which he questioned the emphasis placed on mathematics in our education system, struck a sympathetic chord in mathophobes whose experience of mathematics consisted of hours of blank incomprehension dealing with 'cosines and algebraic abstracts'. He states that for many pupils the 'useless hours spent on Euclid' would have been better spent on "understanding the most basic life-skills: mortgages, money management".*
>
> **John White, Irish Times, 5 June 2003**

> *What will mathematics education look like in 2000? The answer is simple. There will be no more mathematics education in 2000, it will have disappeared. There will be no subject called mathematics, no math programme, no math textbook to teach from… It is there to be lived and enjoyed, just as reading, writing, handicrafts, arts, music, breathing, in integrated education.*
>
> **Freudenthal, 1977, p. 294**

One hundred years ago access to post-primary education was only open to a small elite bound for a career in the civil service or professions. Today in Ireland, three to five years of post-primary education is a shared feature of almost every person's educational experience, with four out of five students who start school going on to complete the Leaving Certificate. In Ireland, mathematics has been and is a core aspect of students' post-primary school experience until the completion of schooling as, unlike some other jurisdictions, students cannot choose to opt out of mathematics at senior cycle. In Northern Ireland, for example, students can choose not to take mathematics as an A level subject, thus completing their post-primary mathematics education at GCSE level - the equivalent of the Junior Certificate. The Northern Ireland pattern is similar to that across the developed world (Le Metais, 2003) where mathematics is not a compulsory subject in upper post-primary education. The fact that mathematics is a part of every Irish student's experience throughout their post-primary school years presents a particular challenge in radically reforming, partially revising or tinkering with mathematics education in order to meet the diverse capabilities and interests of entire student cohorts. In discussing the nature of any proposed reforms in mathematics education, we draw attention to the importance of interlinking these with wider post-primary reform directions and strategies in order to increase the likelihood of economies of scale and synergistic forces than can operate in broadly based systemic reforms.

The current focus on redefining and reforming mathematics education in many countries constitutes a *conjuncture* (Goodson, 2002), that is, a powerful educational movement which sweeps around the world, knowing no respect for national boundaries. In light of these two significant challenges, of redefinition and reform, this final chapter focuses on their implications for post-primary mathematics education. Both from a national and international perspective, these challenges have emerged as urgent issues and are an illuminating barometer of the changing relationship between school and society in an era of globalisation, with its calls for a move toward a knowledge-based society. As we noted earlier, the cultural pressure to redefine mathematics emanates from a variety of sources, including disenchantment with the overly abstract focus of the now longstanding 'new' or 'modern' mathematics curricular culture, alarm among the business community (and some educators) at students' limited capacity to apply knowledge in new contexts, pressure from the learning sciences to revise our deeply held ideas and assumptions about both learning and mathematical understanding, the unprecedented elevation of international comparative test results onto government and cabinet tables, and deep concern about perceived and/or actual gender, socioeconomic status (SES) and ethnicity gaps on mathematics achievement tests. The cultural pressure to redefine mathematics education has led or is leading to significant educational reform agendas in a number of countries/regions (e.g. China; Victoria, Australia; Germany; Singapore; USA; UK).

This chapter addresses the issues raised in this report in terms of their potential to contribute to the current review of post-primary mathematics education in Ireland. The review is the first such opportunity in Ireland for over forty years, and is intended to be a root-and-branch initiative rather than a tinkering with the current model of mathematics at both junior and senior cycle (NCCA, 2005). In earlier chapters, we have sought to provide a wide range of examples from various countries around the world to illustrate how they are grappling with redefining mathematics education. In doing so, we have drawn on debates on mathematics education, mathematics education policy initiatives, new cross-national research on classroom practice using video technologies, recent developments in the learning sciences and mathematics education, and current understanding of educational systems as complex ecologies (Hoban, 2002). Considering the myriad challenges that face any attempted reform of mathematics education, we emphasise that it is highly unlikely that merely changing one feature of the system -the mathematics curriculum culture, textbooks or testing/examinations alone - will result in new forms of mathematics education in Irish post-primary classrooms. Even if all three were to be the focus of reform efforts, it is highly unlikely that actual changes in classroom practice or student scores in international tests would be evident on a wide scale for at least five to ten years. Nevertheless, in writing this chapter we assume the need for some moderate - if not significant - mathematics education reform given Ireland's ranking in international comparative studies vis-à-vis the country's ambitious economic, educational, social and research goals.

This does not necessarily mean that PISA results alone should be used, for example, as a reason for curriculum change in mathematics, but that a careful consideration of the relative merits of adopting a PISA-like approach to mathematics education may have considerable value. The merits of such an approach are not based solely on how a PISA-like reform of mathematics education might deliver a higher ranking in, for example, a PISA study a decade from now, but on how it may change the meaning of mathematical competence and create, in the long term, a more mathematically literate society ready to embrace the challenges of the 21st century. The necessary conditions for a radical review of mathematics education have far-reaching implications, encompassing primary and post-primary teacher education; the teaching of mathematics in tertiary institutions (especially those where future mathematics teachers enrol); the scope, rigour, practice-related and sustained nature of teacher professional development; the design of assessment and examinations; the nature of teaching resources (i.e. textbooks and ICTs); and the assumptions underpinning people's 'everyday' and the system's 'official' conception of mathematics education.

Ireland's current economic and social goals, whether expressed in terms of its desire to become a knowledge economy or for Irish universities to be ranked in the top quarter of OECD country universities, are ambitious ones that present definite challenges in light of existing human capital (of which mathematical literacy can be seen as a component part) and infrastructure. In relation to the knowledge society goal, some would argue that it has already been achieved, whereas others would claim that there is only a vision not the reality of a knowledge society. Whichever is the case, mathematics is typically seen as playing a very important role. For example, the former Chief Science Advisor (McSweeney, 2005) to the Irish government argued that mathematics may be even more central than the hard sciences such as physics (important as they are) in providing an underpinning for the knowledge society. Ernest (2000), reflecting on mathematics education in England, argues that 'the utility of school and academic mathematics is greatly overestimated, and the utilitarian argument provides poor justification for the universal teaching of the subject throughout the compulsory years of schooling' (p. 2). However, he extends his critique by making a case for a more expansive set of aims for school mathematics, encompassing (i) skills and enhancement of knowledge-based capability, (ii) the development of creative capabilities in mathematics, (iii) the development of empowering mathematical capabilities and a critical awareness of the social applications and uses of mathematics, and (iv) the development of an inner appreciation of mathematics, including its big ideas and nature.

However, in considering the role of mathematics in the wider curricular and societal context, neither education systems nor curricular areas derive their full rationale from the needs of the economy alone. It is a marked feature of curricula everywhere that they must balance multiple educational goals. The competing nature of those important goals presents real challenges at system, school and student levels. For example, debates about reforming mathematics education at senior cycle might differ significantly if only a small percentage of each senior-cycle cohort were studying the subject. The fact that mathematics is a subject taken by all students until the completion of their post-primary education means that curriculum review and reform must address both the two most significant mathematics education goals: 'mathematics for scientific advancement' and 'mathematics for all'[15].

The remainder of this chapter is divided into two parts: (i) a brief overview of the post-primary mathematics education context in Ireland; and (ii) a discussion of five key challenges which emerge from the preceding chapters in the context of the current review of mathematics education in Irish post-primary education.

5.2 Context of post-primary mathematics education in Ireland

In this section, we summarise key issues related to mathematics education in Ireland under two headings: (i) concerns about post-primary mathematics education; and (ii) recent research on post-primary mathematics education in Ireland.

Concerns about post-primary mathematics education

Concerns about mathematics education have been in the news in Ireland over the last few years. In particular, the publication of PISA mathematics literacy results (a preliminary report was published in

15 The NCCA's Discussion Paper (Review of Post-primary Mathematics Education, NCCA, October, 2005) on post-primary mathematics education refers to 'mathematics for scientific advancement' and 'mathematics for all' as 'specific' and 'general' mathematics education respectively.

December 2004 and full report in April 2005[16] by the Educational Research Centre) and the Leaving Certificate results in August 2005 drew attention to a number of concerns about mathematics education - some new, some not so new. A perusal of the related newspaper headlines presents a range of concerns reflecting the *push* and *pull* factors found in other countries (see chapter one, section 1.4). In August and September 2005, concerns about mathematics at post-primary level that were expressed in national newspapers partly repeated the annual discussion of exam results on the occasion of the publication of the Leaving Certificate results, but they also reflected a broader disquiet about the subject at post-primary level. Among the headlines were the following:

- **Poor maths results just don't add up**
  (Irish Times, 12 August 2005, p. 14)

- **Overhaul of Leaving Cert Maths urged by key group**
  (Bottom front page, Irish Times, 16 August 2005)

- **High rates of failure in Leaving Certificate maths and science**
  (Main headline on front page of Irish Times, 17 August 2005)

Typically, these concerns are linked, both by politicians and the business sector, to the country's economic development and/or the important role of mathematics in helping create a knowledge economy. Although taking a somewhat wider perspective than mathematics alone, a recent article on Ireland's low output of knowledge-economy ready students claimed that:

> *In recent years students (and their parents) have shown a declining interest in third-level courses that lead to careers in science and technology.*

16 Proceedings of the April 2005 national conference reporting on the PISA 2003 findings were published in September 2005. The proceedings include a summary of papers of key findings in Education for Life (Cosgrove, et al., 2005) followed a by a summary of the feedback from educationalists to the ERC and PISA steering committee. Available online at: www.erc.ie/pisa/

> *At second level, this is reflected in lower take-up of relevant school subjects, as well as lower performance in them. This trend has now reached crisis proportions. (O'Hare, Irish Times, 17 August 2005, p. 14).*

O'Hare's comments reflect a concern that applies to mathematics as well as other high-yield subjects in the development of a knowledge economy. In a similar vein, the Irish Business and Economic Federation's (IBEC) director of enterprise (Butler, *Irish Times*, 12 August 2005, p. 14) contrasted the improvement, between 1994 and 2004, in the percentage of honours students in nine other LC subjects with the decline in the percentage of students achieving a C or greater in LC honours maths: 84 percent got an honour in 1994 compared to 77 percent in 2004. He elaborated by arguing that the number of students not passing ordinary level mathematics (approximately 5,000 in 2005), combined with the decrease in interest in science subjects, 'should ring alarm bells around the cabinet table, not just in the Department of Education'. Furthermore, he warns that, given the Industrial Development Authority's (IDA) prediction that biotechnology, a mathematics-dependent sector, will be the next big driver of Ireland's knowledge-based economy, what is especially worrying is the mismatch between stated government aims of fostering a knowledge society and the fact that fewer students are 'leaving school with the skills and interest needed to support the projected growth of these sectors'. Stressing the vital role of mathematics, he claims that, 'virtually no quality career… will be available in high-tech sectors without a high-level knowledge of maths'. Three days later, in a front-page article in the *Irish Times*, IBEC again expressed concerns about mathematics, claiming that, 'The current curriculum has failed to give students a good understanding of the practical uses of maths outside of the classroom and must be addressed' (*Irish Times*, 17 August 2005, p. 1).

The above concerns reflect a mixture of *push* factors (that is, a perceived declining standards and poor capacity to apply mathematics in real world contexts) and *pull* factors (that is, the likely mathematical needs of the economy in newly emerging areas of science that will demand sophisticated knowledge and use of mathematics in problem-solving contexts). While the emphasis is clearly on 'mathematics for scientific advancement', the concerns being raised about foundation level mathematics reflect a wider concern about the mathematics capacity of more than just an elite group of high-flying high-tech-bound mathematics students. As such, there is some evidence of a 'mathematics for all' emphasis in some of the concerns currently being raised about mathematics in Ireland. However, the 'mathematics for all' argument, as used in mathematics education, typically focuses not only on mathematics for economic purposes, but also its role in educating a critically informed, numerate citizenry.

To the extent that media debates about mathematics focus on a restricted version of 'mathematics for all', it is likely that some important functions of mathematics in society may be overlooked. Scribner's three-metaphors approach to characterising literacy (1986) is helpful in this context: that is, literacy as adaptation, literacy as power, and literacy as a state of grace. While she was referring solely to reading literacy, she stressed the important role of each type of literacy. The *literacy* as adaptation metaphor focuses on how reading literacy or mathematics literacy allows the person to survive and adapt in society. Note, though this is more than just 'basic survival' and can also include high-level adaptation, the emphasis is nevertheless on the adaptive, survival function of literacy within existing economic and political structures. *Literacy as power* stresses the political empowerment that can accrue to people through access to literacy. In a similar fashion, one might ask in what way might access to mathematics enhance people's capacity as citizens to transform the world around them. Finally, Scribner writes about *literacy as a state grace*: that is, a form of individual or collective access to literacy's privileged forms of power, knowledge and aesthetic experience. With different but related emphases, the multi-dimensionality of literacy is evident in the OECD's expansive definition of mathematics literacy in PISA:

> *…an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage with mathematics in ways that meet the needs that of the individual's life as a constructive, concerned and reflective citizen (OECD, 2003, p. 24)*

The OECD's definition of mathematics is based on its literacies framework rooted, as we have noted earlier, in socio-cultural understandings of language and literacy.

Specifically, PISA drew on Gee's (1998) definition of language and the PISA studies extend Gee's understanding of language and its allied framework to view mathematics as a language in its own right, in terms of people's knowledge of both its design features (facts, signs, symbols and procedures) and the potential uses of these resources in a variety of non-routine situations for different social functions (see OECD, 2003, p. 26) (see also RME in chapter three and Mathematics in Context, that is, case one in chapter four). The adoption by the OECD of this socially embedded model of knowledge in PISA represents a very significant move toward acknowledging that understanding the use or application of knowledge is not merely a curricular afterthought, along the lines of, 'nice if we can manage to fit in some applied bit of maths but real maths is more abstract!'

It ought instead to be a constituent part of how teaching and learning are conceptualised, in order to support knowledge building by drawing upon a range of cultural and psychological tools and resources: classroom social interaction and support, technologies, symbols, and non-routine problems. The expansive definition of mathematical literacy in PISA, given its focus on the development of constructive, concerned and reflective mathematically literate citizens, presses for a vision of mathematics beyond one hitched only to economic imperatives.

Irish 15-year-olds' mathematical literacy scores: crisis or no crisis?

The process of making claims about the extent and nature of a crisis, if any, depends on appraising the meaning of a number of different types of test results in the light of educational and societal goals. Results of tests typically report two types of scores: measures of central tendency (known as the 'average' in our everyday use of that term but referred to as the 'mean') and measures of variation or distribution of scores. In the case of PISA, a country's mean score is used to create a ranking of where a country lies in relation to the OECD average and other countries - what some have called a 'cognitive Olympics' approach to education. Measures of dispersion provide an indication of how student scores differ from the mean.

The moderate scores of Irish 15-year-old students on PISA mathematics literacy results have drawn unprecedented attention to, not only the actual ranking of Irish students, but also to an emerging debate in the Irish mathematics education and wider education community about the nature of post-primary mathematics. Close and Oldham (2005), commenting on the 'marked contrast' between Irish students' high ranking in reading literacy (5th of 39 countries) and their mid-ranking in mathematical literacy (20th of 40 countries), warn that:

> *The mathematics performance is of some concern since mathematics is considered a key area of competency in moving towards a knowledge-based society and economy. In this context, the contrast between the type of questions in the PISA mathematics tests and those typical of the Irish Junior Certificate examinations has been the subject of discussion. (p. 174)*

However, given the different philosophies underpinning the Junior Certificate and PISA mathematics literacy, one could argue that the performance of Irish 15-year-olds was higher than expected: i.e. Irish students performed at about OECD average (Cosgrove, Oldham and Close, 2005). Overall, Ireland's scores in mathematical literacy resulted in a ranking of 17th of 29 OECD countries. In terms of the distribution of scores, Ireland had 'comparatively few very high achievers and very low achievers'. That is, 17% of Irish students achieved Level 1 or below Level 1 (the lowest two levels on the proficiency scale), compared to an OECD average of just over 21%, whereas approximately 11% of Irish students achieved Levels 5 and 6 (the highest levels), compared to an OECD average of 15% (Cosgrove *et al.*, 2005). The comparatively low level of very high achievers raises concerns about whether current post-primary mathematics education is preparing a sufficient number of students to meet the demands of a knowledge society which will need a pool of researchers and other professionals for a range of fields underpinned by mathematics, as well as mathematics graduates who may enter the teaching profession. One other indicator of the pattern of Irish scores is particularly informative. The average difference in

Ireland (220.9) between the 10th and 90th percentile scores was lower than the OECD average difference (259.3), providing evidence that there is a narrower band of achievement in Ireland than in many other countries (Finland's and Korea's were also small). This can be seen as a positive outcome in that the Irish education system provides, relative to most other OECD countries, a more equitable outcome in mathematical literacy. Cosgrove *et al*. (2005) note that the combination of high achievement and a narrow spread of achievement in Finland and Korea is important in demonstrating that high average achievement scores and low variation in achievement scores are not incompatible. In chapter two we noted a similar pattern in the case of Japan's TIMSS (1995) results and that this pattern was one of the main reasons for the attention accorded Japanese mathematics education over the last decade. In summary, the ranking based on the mean scores raises serious questions about Ireland's overall performance; the pattern of dispersion raises questions about the dearth of very high achieving students; and the narrow spread of achievement is a welcome finding.

Is there or is there not a crisis in mathematics education? On the one hand, there is the no-crisis argument that the existing mathematics education system has provided a sufficient pool of mathematically competent people to support the Celtic Tiger economy. On the other hand, Ireland's mid-ranking PISA scores may not be sufficient to meet the social and economic aspirations in today's Ireland. Overall, given the ambitious social, economic and research goals being set by the government and research agencies and institutions, we think that the mid-ranking scores and the comparatively low level of very high achieving students are a cause for considerable concern. To the extent, as Dewey (1938) has argued, that education is about providing experiences with which students can grow into more fruitful and expansive engagements with future experiences (even if the nature of these is as yet somewhat unclear), it is important, at the very least, to consider carefully the nature of post-primary mathematics education. The decision as to whether a reform of post-primary mathematics education might move toward a PISA-like approach to mathematics is inextricably linked with a vision of what the competent mathematical learner of 2010 and 2020 ought to look like. That in turn is based on our incomplete knowledge of how society will evolve in the next few decades and what that will mean for mathematical ways of knowing.

Recent research on mathematics in Ireland

There has been a considerable amount of research on mathematics education at primary and post-primary levels in Ireland over the last five years which can contribute in important ways to the current review of post-primary mathematics. In light of Lyons *et al.*'s (2003) observation that over the last thirty years there has been 'a limited amount of research undertaken on mathematics education in the Republic of Ireland' (p. 2), the recent publication of their landmark study and two other major publications (Cosgrove *et al.*, 2005; Close *et al.*, 2005) on mathematics education this year represents a significant development. In this section, we summarise major findings rather than review the research. Three recent publications bring together much of the relevant research: *Inside Classrooms: the Teaching and Learning of Mathematics in Social Context* (Lyons, Lynch, Close, Sheerin and Boland, 2003); *Education for Life: The Achievements of 15-year-olds in Ireland in the Second Cycle of PISA*[17] (Cosgrove, Shiel, Sofroniou, Zastrutzki and Shortt, 2005); and the *Proceedings of the First National Conference on Research in Mathematics Education* (Close, Dooley and Corcoran, 2005).

17 Education for Life (Cosgrove et al., 2005) can be downloaded at the following web address: www.erc.ie/pisa/

Inside Classrooms involved a video study of junior-cycle mathematics classes in coeducational and single-sex post-primary schools. It is a landmark study in Irish education in that it is the first video study of classroom teaching in Ireland, and provides insights into pedagogy and its impact on student attitudes to mathematics and their opportunities to learn. The *Inside Classrooms* study had a specific focus on questions of gender and its relationship to classroom learning opportunities in mathematics. In line with the video studies discussed in chapter two of this report, it reflects a move toward a contextual and up-close analysis of classroom teaching.

Education for Life (Cosgrove *et al.*, 2005) provides a detailed analysis of the PISA 2003 results, in which mathematical literacy was a major domain, for Irish 15-year-olds, as well as a detailed account of the PISA mathematical literacy framework (Cosgrove *et al.*, 2005, pp. 4-11).

In the context of the current review, the 300-page-plus *Proceedings of the First National Conference on Research in Mathematics Education* provides a timely and detailed set of twenty-three papers, many of which are directly relevant to post-primary mathematics education.

What shapes student experiences of maths? Curriculum culture, textbooks and examinations

In earlier chapters, we focused on three system-level features which profoundly shape students', teachers' and parents' experiences of mathematics education: curricular culture, textbooks, and testing/examination traditions. We now outline relevant research in Irish post-primary mathematics education pertaining to these features.

The curricular culture of Irish post-primary mathematics education

The *Inside Classrooms* video study provides convincing evidence[18] that Irish post-primary mathematics education at junior cycle is dominated by a culture of teaching and learning 'which presents the subject as static rather than dynamic, abstract, formal and remote rather than relevant and accessible' (p. 363). The video study documents the close relationship between teachers' stated beliefs about teaching and learning mathematics and their actual classroom practices (p. 366). When interviewed, teachers stressed the importance of the 'demonstration of procedures and monitoring of students' progress and the video analysis bore this out' (p. 366). This finding reflects well-established findings in research on teacher thinking which has demonstrated a considerable degree of consistency between teachers' stated views on teaching and their classroom practices (Clark and Peterson, 1986; Morine-Dershimer, 1997). Lyons *et al.* conclude that:

> *…all twenty lessons that we videotaped were taught in a traditional manner… most of the time was spent on exposition by the teacher, followed by a programme of drill and practice. Overall teacher-initiated interaction comprised 96% of all public interactions in the classes, and within this context a International Trends In Post-Primary Mathematics Education procedural rather than a conceptual and/or problem-solving approach to subject matter prevailed. (p. 367)*

Oldham (2002) anticipated these findings in her comments on the historical roots of Ireland's post-primary mathematics education culture:

> *At second level, Ireland adopted 'modern mathematics' in the 1960s to greater extent than many other countries (Oldham, 1989; Oldham, 2001); the syllabuses and, in particular, the examination papers,… still*

18 A single class group for two class periods was videoed in ten different schools. While the sample of lessons and teachers was small, and not nationally representative, we think the video study is convincing, given the relative homogeneity of the education system in Ireland.

reflect the focus on precise terminology and abstraction that is characteristic of the movement. The recent revision of the junior cycle syllabus was less profound than the revision at primary-school level; it was essentially a minor update to deal with areas of the course that were giving difficulty rather than a root-and-branch review. (p. 43)

The video study evidence and historical context both suggest that the mathematics education culture existing in Irish post-primary schools (at junior and senior cycles) is didactic and procedural, and as such consistent with the new mathematics education movement. Lyons *et al.* go as far as to say that, based on evidence of curricular cultures in other subjects at second level, 'the evidence from this study is that mathematics classes are especially didactic' (p. 367).

Textbooks in Irish post-primary education

Whereas the *Inside Classrooms* video study and the historical context of Ireland's adoption of new/modern mathematics is now well documented, there is a dearth of evidence on the nature of post-primary mathematics textbooks. What mathematics education curricular culture informs and is reflected in the textbooks? What is the role of examinations in shaping textbook content and format? What types of mathematical ways of knowing are privileged in post-primary mathematics textbooks? From the PISA competency clusters perspective, to what extent do textbooks focus on each of the three clusters, namely, reproduction, connections and reflection? To use PISA terminology, what is the range of 'contexts' provided in presenting mathematical ideas? These and other questions are important in considering the role of textbooks in any effort to redefine post-primary mathematics education. Some tentative findings in the Irish context may be gleaned from the TIMSS textbook analysis (see chapter one, section 1.5). Internationally, the mismatch between stated mathematics education reform aims and the marked dearth of textbook content reflecting these reform aims, identified in the TIMSS textbook analyses, in which Irish textbooks were included for two of the three populations, suggests that it is unlikely that Irish post-primary textbooks are any different. Indeed, scanning the textbooks it appears that they, like the Junior and Leaving Certificate examinations, rely on a similar pattern of exposition as evidenced in examination papers: that is, a focus on precise terminology, symbol manipulation and abstraction with little attention devoted to the provision of rich contexts to concretise mathematical ideas. From a Realistic Mathematics Education perspective, we think it is reasonable to argue that textbooks at post-primary level typically focus on vertical mathematising[19] rather than providing students, teachers and families with opportunities to experience the full horizontal and vertical mathematising cycle. In the context of the review, we note that one of the features of RME-influenced reforms in the Netherlands was a revision of textbooks in order to create contexts in which teachers and students could grapple with mathematical ideas in ways consistent with the horizontal and vertical mathematising cycle underpinning RME.

Testing/examination traditions in mathematics at post-primary level

> *They were only weeks away from the examination halls. So much work still had to be done, so much work had to be gone over again. The chance-throw of the exam would almost certainly determine the quality of the rest of their lives. Sheila had dreams of university. Much could be won, a great deal could be lost, and there was always England.*
>
> **(John McGahern, Amongst Women, pp. 72-73).**

19 Horizontal mathematising refers to the process of taking real-life situations and turning them into mathematical models. Vertical mathematising refers to the manipulation of symbols and advancement of mathematical modelling and/or translating this back to the horizontal level. The horizontal-vertical distinction has its origins in RME (see chapter three). A good example of failed translation of mathematical models back to the horizontal is how IMF and World Bank economists (see Stiglitz, 2003) are often accused, unfairly or not, of failing to understand the implications of their econometric model-based prescriptions on real-life economics in vastly diverse settings, where culture, or other important variables, cannot be reduced to a specific amount of variance in multiple regression models. Such arguments and criticisms draw attention to the hermeneutic dimension of translating mathematical models of social and economic dynamics to real-life situations.

The centrality of the examination tradition in Irish post-primary education provides a good opportunity to examine the type of knowledge valued in any subject area in Irish education. In the case of Irish post-primary mathematics, Close and Oldham (2005) have recently undertaken a comparative content analysis of certificate examinations and the PISA framework, in which they mapped the 2003 Junior Certificate and the 1974 Intermediate Certificate onto the three-dimensional (content, competencies and situations) PISA mathematical literacy framework. Their study is illuminating on two levels. Firstly it provides insights into the curricular culture of post-primary mathematics and reveals the consistency across time of a particular view of mathematics that is aligned with the 'new' mathematics movement. Secondly it reveals striking differences between the type of mathematical knowledge that was required for success on both the 2003 Junior Certificate and 1974 Intermediate and that in the PISA mathematical literacy framework. As Close and Oldham observe:

> *…major differences can be seen between the PISA tests and Irish papers. There are considerable discrepancies across specific Irish papers in terms of the percentages of items in two of the three competency clusters (Reproduction and Connections), again reflecting the levels of the examinations. (p. 185)*

They also identified a 'major difference' in the situations component between the PISA framework and examination papers. The Close and Oldham (2005) and Cosgrove *et al*. (2005) studies draw attention to the significant discrepancies between PISA and Irish post-primary mathematics. It is important to note that PISA mathematical literacy assessments were never intended to reflect or assess national mathematics syllabi; rather PISA mathematical literacy is based on a vision of mathematical competence that PISA views as necessary for students' life as productive and reflective workers and citizens in the 21st century. The Close and Oldham study, in particular, draws attention to how the examination tradition emphasises a certain type of mathematical competency (i.e. reproduction) and draws upon narrow range of mathematical situations (i.e. mainly intra-mathematical).

Table 7: Percentage of items in the 2003 Junior Certificate and 1974 Intermediate Certificate mathematics papers corresponding to the three clusters of the PISA mathematics framework

PISA framework dimension	Dimension category	PISA 2003 maths test (85 items) % of	2003 JC Higher level maths (77 items) % of	2003 JC Ordinary level maths (81 items) % of	2003 JC Foundation level maths (32 items) % of	1974 JC Higher Course maths (79 items) % of	1974 IC Lower Course maths (79 items) % of
Competency clusters	Reproduction	30.6	83.1	95.1	100	84.8	88.2
	Connections	47.1	16.9	5.0		15.2	11.8
	Reflection	22.4					

SOURCE: Close and Oldham, 2005, p. 183

Table 8: Percentage of items in the 2003 Junior Certificate and 1974 Intermediate Certificate mathematics papers corresponding to the context categories of the PISA mathematics framework

PISA framework dimension	Dimension category	PISA 2003 maths test (85 items) % of	2003 JC Higher level maths (77 items) % of	2003 JC Ordinary level maths (81 items) % of	2003 JC Foundation level maths (32 items) % of	1974 JC Higher Course maths (79 items) % of	1974 IC Lower Course maths (79 items) % of
Situations	Personal	21.2		12.3	21.9	1.2	13.2
	Educational Occupational	24.7	6.5	23.4	6.2	7.6	7.4
	Public	34.1	16.9	2.5	12.5	8.8	10.3
	Scientific	20.0	10.4				
	Intra-mathematical		66.2	61.7	59.4	82.2	69.1

SOURCE: Close and Oldham, 2005, p. 184

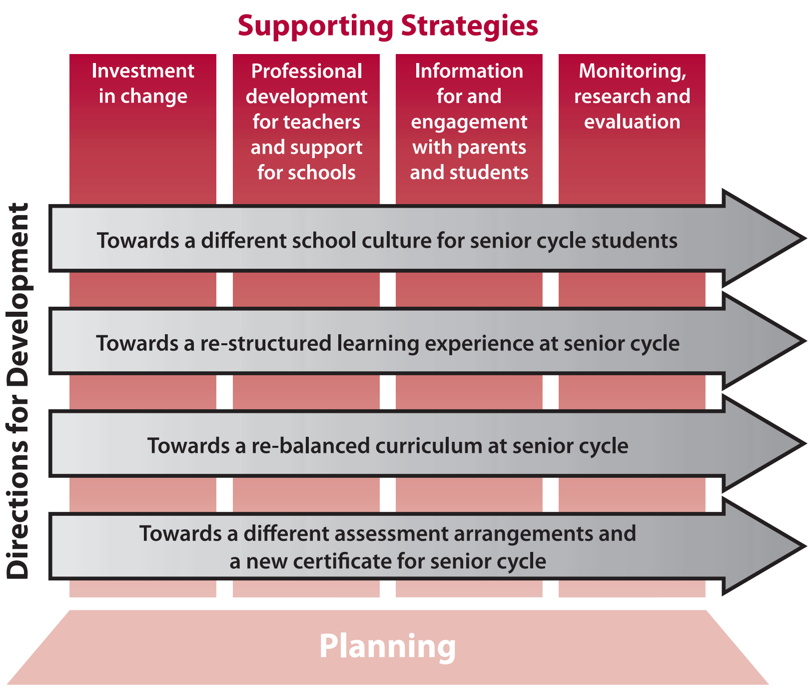
The intra-mathematical dominance of context (situations) illustrates how the abstract focus of new mathematics has filtered into the type of examination questions likely to be asked, and thus likely to guide teachers and students in their exam preparation (see table 8) (Elwood and Carlisle, 2003). Elwood and Carlisle (2003, pp. 72-74), in their examination of Leaving Certificate mathematics papers (they examined both content and question type) found that questions in the Leaving Certificate were abstract, focused on 'pure' mathematics, relied more heavily on traditional notations and symbols than comparable examination papers in other jurisdictions, and were notable also for the lack of real-life contexts in questions. In the next section we note key reform challenges in mathematics in the light of the wider reform context at junior and senior cycle in Irish post-primary education.

5.3 Five challenges within a wider post-primary reform context

> *…systems thinking is a discipline for seeing the 'structures' that underlie complex situations…systems thinking offers a language that begins by restructuring how we think.*
>
> **(Senge, 1990)**

Post-primary education in Ireland has been the focus of considerable attention over the last few years, due largely to the acknowledgement that while the curriculum and examination system has served society well it may not be providing students with the necessary learning experiences for the 21st century, especially in an emerging knowledge society. The National Council for Curriculum and Assessment proposals for senior cycle sets out possible reforms for upper post-primary education (NCCA, 2003; NCCA, 2005). The wider reforms outlined by the NCCA set a context for any proposals to reform post-primary mathematics education. Developing a different school culture, reforming students' experiences of teaching and learning, re-balancing curriculum at senior cycle, and developing different assessment arrangements and a new certification system for senior cycle, were the directions for development specified in the NCCA document (see Figure 6). The documents outline a series of support strategies that would be necessary to achieve these ambitious goals, encompassing investment for change, professional development for teachers and support for schools, provision of information to and engagement with stakeholders and parents, and monitoring, research and evaluation. Given the interlocking nature of the forces impacting teaching and learning, any effort to reform post-primary mathematics education will have to adopt this or another similarly comprehensive systemic approach.

Figure 6: Proposals for the Development of Senior Cycle Education Directions for Development and Supporting Strategies



The wider post-primary review and reform context is important in considering the potential for serious reform of mathematics education. Roschelle *et al.* (2000), for example, in a comprehensive review of ICTs and learning, observed that technological innovation in education needs a wider reform context for ICTs to have significant impact on teachers' classroom practice and students' learning (for a discussion ICT innovation and educational change in the Irish context see Conway, 2005c; Fitzpatrick and Conway, 2005). Similarly, we want to draw attention to how any effort to reform mathematics education will be supported and/or inhibited by the wider post-primary reform context. For example, should mathematics education adopt a more problem-solving, realistic mathematics education (RME) approach, its impact might be significantly constrained by students' experiences of mathematical ideas in other subjects. If mathematics is presented in subjects such as physics, economics, and geography only as the application of procedures, it might detract from their sense of how both horizontal and vertical mathematising provides a powerful means through which students can engage with the world. In essence, there has to be congruence between students' experiences of mathematical ways of thinking and practices across different subjects. Increasingly, over the last decade, ecological and systems perspectives of educational change have gained prominence over linear, additive and mechanistic change models (Hoban, 2002). Ecological and systems perspectives draw attention to the inter-relationship of a variety of forces and how they might impact educational change: school culture, teachers' lives and learning, assessment systems, teaching and other resources, everyday/'folk' conceptions of teaching and learning, politics, leadership and parental/societal expectations. From a systemic and ecological perspective, any effort to reform mathematics education without attending to the wider context is highly unlikely to bring about the desired changes.

The cumulative impact of the interlocking forces of a curriculum culture, textbooks and examinations mutually reinforcing each other cannot be underestimated in terms of the challenge it poses for any proposed reforms in post-primary mathematics education. The challenges that might be addressed can be outlined under five headings:

- defining a vision
- changing examinations and assessment practices
- the excellence/equity tension
- teacher education
- scaling up: the change challenge.

Defining a vision for mathematics education today

Any claims about maintaining the status quo, engaging in minor revisions, or making radical reforms of mathematics education reflect 'values, interests and even ideologies of certain individuals and groups' (Ernest, 2000, p. 5) Ernest provides an overview of interest groups and their aims for mathematics teaching in Great Britain (see table 9). There is considerable resonance between the aims and locations described by Ernest and aims and locations in the Irish context. Consequently, it is important to consider the value-laden nature of any claims made in relation to the state of mathematics, the interpretation of evidence in discussing the state of mathematics in schools, the discursive space accorded various publics (or stakeholders), and the scope, focus and appropriateness of current and potential aims for mathematics education. For example, to what extent do all students need maximal mathematical knowledge? Is basic numeracy sufficient for a large majority of students? Should appreciation of the big ideas of mathematics be a feature of any, all or just some students' mathematics education? To what extent ought mathematics education focus on a critical appreciation of the social applications and uses of mathematics?

Table 9 Interest groups and their aims for mathematics teaching

Interest group	Social location	Mathematical aims
Industrial trainers	Conservative politicians	Acquiring basic mathematical skills and numeracy
Technological pragmatists	Meritocratic industry-centred managers	Learning basic skills and learning to solve practical problems with mathematics and ICTs
Old humanist	Conservative mathematicians preserving rigour of proof and purity in mathematics	Understanding and capability in mathematics with some appreciation of mathematics (pure mathematics and/or 'new' mathematics-education centred)
Progressive educators	Professionals, liberal educators, welfare state supporters (child-centred progressivist)	Gaining confidence, creativity and self-expression through mathematics
Public educators	Democratic socialists and reformers concerned with social justice	Empowerment of learners as critical and mathematically literate citizens in society

SOURCE: Ernest, 2000, p. 6

Is the development of students' skill or capability in mathematics necessarily competing with the development of students' awareness of big ideas in mathematics? To what extent should some or all students experience horizontal and vertical mathematising (see chapter three)? Ought school mathematics focus first on teaching basic skills prior to teaching higher-order conceptual understanding in mathematics? It is probably important to address some or of all of these - and other - questions in redefining post-primary mathematics education, aware that answers to each of these questions inevitably reflect value positions, some more amenable to empirical investigation than others, but each important nonetheless. Different sectors in the education system (primary, post-primary, tertiary), government, industry[20], business and the wider society have legitimate interests in and claims on how mathematics is and could be taught in post-primary education, and there will be differences in how each might respond to these questions because of their different interests.

In this report, we have emphasised the powerful role of the mathematics curricular culture, textbooks, and examination/assessment traditions, each with identifiable assumptions and values in relation to mathematics education, particularly the nature of what counts as worthwhile mathematical knowledge. In many ways, these can be thought of as interlocking forces - each shaping and reinforcing the other over time. Of these three system-level features, curricular culture is probably the most difficult to change as it is threaded through other aspects of mathematics education, including textbooks, examination and assessment

20 For example, the Irish Council for Science, Technology and Innovation (ICSTI), identified effective primary and second level science, technology and mathematics (STM) education in Irish schools as a priority area for its consideration, and undertook a benchmarking study titled Benchmarking School Science, Technology and Mathematics Education in Ireland against international good practice (ICSTI, 1999). Available online at: http://www.forfas.ie/icsti/statements/benchmark/foreword.htm

traditions, and teachers' and students' mathematical beliefs and practices in the classroom, as well as parents' understandings of mathematics as they support their post-primary students' mathematics learning. Questioning epistemologies, examining philosophies of mathematics education, and reforming teaching and learning practices are central to developing any new vision of post-primary mathematics education. These aspects of reform have implications at system, school and classroom level; we address them in general terms, noting issues of relevance to teacher education (initial, induction and in-career) as teachers' opportunities to learn will be central in any proposed reform of mathematics. For example, if a culturally appropriate adaptation of lesson study was to be implemented it would have significant implications for investment in teacher education at all phases in the teaching life-cycle.

Questioning epistemologies

Underpinning any discussion of curricular culture lies a conception of knowledge in the particular domain. These conceptions are reflected in the personal epistemologies of all involved (teachers, students, parents, curriculum and test/examination designers) and encompass questions about certainty of knowledge in the domain, the basis for making knowledge claims, and the organisation of knowledge in the domain (Hofer, 2004). In this report, we address epistemology through our description of the differing epistemologies underpinning behavioural, cognitive and constructivist approaches to learning.

The realistic mathematics movement adopts a phenomenological epistemology: that is, it focuses on the lived mathematical experiences of learners as the basis for mathematising. On the other hand, new/modern mathematics education's epistemology regards mathematics as an 'objective, value-free, logical, consistent and powerful knowledge-based discipline which students must accept and understand and manipulate' (Burton, 1994, p. 207). The 'real world' and problem-focused approaches to mathematics, inspired by Piagetian constructivism, situated cognition and realistic mathematics education, have been more attentive to questions of epistemology. Although we note that Freudenthal (1991) was reluctant to get drawn into discussions of 'isms' in mathematics education and was deeply sceptical of constructivism, which he viewed as slogan ridden (Freudenthal, 1991, pp. 142-46; Goffree, 1993), he nevertheless adopted an epistemology with a family resemblance to constructivism, given RME's phenomenological view of the learner and learning.

Olson and Bruner (1996) argue that any innovation in education must engage actively with the everyday/folk theories of learning held by teachers and educators. In chapter three, we noted two studies which demonstrated the important role of teachers. In chapter three, we also outlined the role that epistemologies play in shaping teachers' classroom teaching practices (Morine-Dershimer and Corrigan, 1997; Staub and Stern, 2002).

In order to highlight the challenge of addressing educators' everyday theories of learning, we draw on Morine-Dershimer and Corrigan (1996) who, having examined twenty years' research on teacher thinking, provide important insights for those promoting reform of mathematics education:

> *The strength of traditional prior beliefs, reinforced by experiences as students and teachers, makes real change extremely difficult. Teachers implementing mandated changes interpret those mandates through the*

screen of their prior beliefs, modifying… desired reform strategies. New practices require new beliefs. Changing beliefs involves cognitive stress, discomfort and ambiguity. In changing beliefs, individuals must reconcile or realign other related beliefs to resolve conflicts and contradictions, and come to terms with what actions guided by previous beliefs meant. Such cognitive reorganization is not easily or quickly accomplished. (p. 308)

They outline three strategies and four conditions for changing teacher beliefs, noting that sometimes teacher practices create changes in beliefs and other times vice versa. The strategies for changing beliefs (these could be thought of as strategies for initial teacher education and/or continuing professional development) are: changing images via exploration of teachers' images and metaphors for teaching; confronting contradictions; and addressing cases. The four conditions for change in teacher beliefs are: time, dialogue, practice and support. They pose an equally daunting challenge in the design and provision of high-quality teacher education. The necessary conditions for changing teacher beliefs share remarkable similarities with key dimensions of Japanese lesson study (see chapter two).

Examining philosophies of mathematics education

The different philosophies underpinning the current approach to post-primary mathematics education and the PISA mathematical literacy framework have helped to highlight the existing culture of Irish post-primary mathematics. Indeed, the high visibility of the PISA results draws attention to the quality of mathematics education in schools, even if only temporarily. In the context of the current review of post-primary education, it is fortunate that the focus on mathematical literacy as a major domain in PISA 2003 (rather than minor domain as was the case in PISA 2000 and as will be in PISA 2006) has fixed a degree of educational and media attention on mathematics.

The differing philosophies underpinning PISA[21] and the 'new' mathematics-influenced post-primary syllabi have inspired two curriculum mapping exercises in Ireland: (i) a test-curriculum rating involving a measurement of the expected curriculum familiarity students might have with PISA concepts, contexts and formats, based on an analysis of Junior Certificate papers at Higher, Foundation, and Ordinary levels (Cosgrove, Oldham and Close, 2005); and (ii) the mapping of the 2003 Junior Certificate and 1974 Intermediate Certificate against the PISA three-dimensional framework (Close and Oldham, 2005). Cosgrove *et al.* (2005) found that 'some key topic areas of sets, geometry, and trigonometry are not assessed at all by PISA items. There is also little coverage in PISA of algebra, and functions and graphs' (p. 203). Such discrepancies help to highlight the consequences of, for example, reorienting syllabi at post-primary level toward concepts that are the focus of PISA mathematical literacy. Debates focused only on whether one topic or another ought to be on future syllabi will be unlikely to provide the type of root-and-branch review of mathematics education currently under way. In order to clarify the curriculum components that might be addressed, the PISA mathematics components (see table 10) may be useful in ensuring broad-ranging discussion. A discussion of the relative strengths of the current post-primary mathematics syllabi using the PISA components (as has already been initiated by the curriculum mapping exercises noted above) provides a useful set of questions in redefining mathematics education at post-primary level.

[21] The OECD's PISA tests are not without their critics. For example, in England there has been considerable debate about the technical adequacy of PISA, the implications sampling of an age group rather than a class group [i.e. PISA sampled 15-year olds rather than a class/year-group as in the case of IEA studies] and PISA designers' lack of attention to linking PISA tests with other international comparative tests in mathematics such as those devised by the IEA (see Prais, 2003; Adams, 2003; Prais, 2004).

Table 10: Mathematics components in PISA and post-primary mathematics

Mathematics component	PISA	'New' mathematics-inspired Irish post-primary mathematics	Reformed post-primary mathematics education
Contexts (Situations)	Diverse range of contexts including personal, educational/ occupational, public, scientific, and intramathematical	Dominated by intramathematical contexts: 60-80% + in Junior Cert (see Close and Oldham, 2005)	What range of contexts will the syllabus include and what will be the emphasis on each in curriculum and assessment?
Content (Overarching Ideas)	4 strands: Space and Shape Change and Relationships Quantity Uncertainty	Syllabi are constructed around curricular strands rather than focused on overarching ideas	What approach to defining content will the syllabus take?.. curricular strands... overarching ideas... other?
Competency Clusters	Reproduction Connections Reflection	Dominated by reproduction (80% + in Junior Cert) but with some focus on connections (see Close and Oldham, 2005)	What types of, competencies, if any, will be expected of students in the new syllabus?

We use the PISA mathematical literacy components not because PISA's version of these components is the only way ahead in curriculum reform, but rather because the components provide a useful set of questions that, we think, are worth addressing in reviewing the post-primary mathematics education. That is to say:

- CONTEXTS: What range of contexts will the syllabus include, and what will be the emphasis on each in curriculum and assessment?

- DEFINING CONTENT: What approach to defining content will the syllabus take:
  - curricular strands
  - over-arching ideas, or
  - other?
- DEFINING COMPETENCIES: What types of competencies, if any, will be expected of students in the new syllabus?

In this section we have noted key dimensions that might be addressed in the redefining of post-primary mathematics education, including discussions of underlying epistemologies, the important role of personal epistemologies for all those engaged in design, enactment and evaluation of mathematics education, and the value of adopting the components of mathematics in PISA as a way of addressing contexts, competences and content in the review process.

In examining post-primary mathematics education, developments at primary level in Ireland provide some useful resources as well as a challenge in terms of how the post-primary system might provide continuity with the more problem-solving and real world focus that has been a feature of primary mathematics - especially over the last six years, since the advent of the Revised Primary School Curriculum (1999) (Conway, 2005b; Coolahan, 2005), and to a lesser extent since the Deweyan- and Piagetian-inspired constructivist orientation embedded in 1971 Primary School Curriculum (see Department of Education, 1971; and for a review Gash, 1993). At present there is distinct discontinuity between philosophies underpinning primary and post-primary mathematics, although the revised junior cycle mathematics makes some moves toward a more problem-solving and constructivist orientation with its focus on fostering students' active learning in mathematics. However, the move towards a problem-solving orientation was reflected more in the Guidelines for Teachers and in-service support for teachers than in the syllabi themselves.

To what extent ought a review of mathematics concern itself with mathematics as used in other curricular areas? For example, the Science, Technology and Society (STS) movement in science education is a kindred spirit to the Realistic Mathematics Education movement in mathematics education. In both movements, there is a range of common assumptions and preferred teaching methodologies, including approaches to learning (both value social support and ICTs in the design of learning environments), the importance of teaching basic skills in the context of real and higher-order activities, and the dual role of reality as both a key source of ideas and an arena for application of models. Recognising the substantial overlap in curricular cultures is an important stance in an ecological approach (Hoban, 2002) to thinking about the future of schools and curriculum reforms. Is it important, in moving toward a knowledge society, that curricular cultures be aligned so that from the perspective of students' educational experience they get a coherent vision of mathematics and learning? So, for example, what are the short and long-term consequences for student learning, if in reforming post-primary mathematics it is poorly aligned with how science, geography or economics teachers teach and use mathematical ideas? For society and the student, the ultimate result may be a weaker grasp of the potential of mathematics and its role as a powerful way of thinking and an important human activity with relevance both in 'real life' and the symbolic world of abstract mathematical modelling. We pose the following thought experiment: what would we learn about one student's experiences of mathematical ways of knowing by shadowing the student for one week across all classes in which mathematics is used and/or taught? Will we learn that this amalgam of experiences leaves the student better off in the sense of being better equipped to think about mathematics? Will the student experience both the real world of mathematical modelling (horizontal mathematising) and symbol manipulation and modelling (vertical mathematising) in each mathematical experience across curricular areas? Will the student get mixed messages about what it means to think mathematically? If yes, what are the consequences, both for the student and for society?

Changing learning, assessment and examination practices

> *Assessment should be recognised, not as a neutral element in the mathematics curriculum, but as powerful mechanism for the social construction of mathematical competence*
>
> **(Clarke, 1996, p. 327)**

> *…the issue now seems to be whether policy makers have both the wisdom to demand that we apply what we already know to design a program that maximizes the benefits and minimizes the negative consequences and the patience it takes to get that job done. In designing such a program, policy makers must realize that, while the effects of tests are often portrayed as uniformly good or bad, tests affect different types of students quite differently and that any particular affect is often a two-sided coin*
>
> **(Madaus, 1991, p. 226)**

Over the last decade there has been a significant realignment of the relationship between learning and assessment (see chapter three). Largely due to very significant developments in both the learning sciences and measurement theory, there are now new ways of thinking about learning and assessment not only as loosely linked ideas, but as closely intertwined but distinct and powerful ways of understanding schooling. For example, developments in self-regulated learning have brought prominence to issues of self, peer and teacher-assessment and how each might support students' ownership of learning. Furthermore, the powerful impact of feedback to learners, alongside clearly articulated domain-specific performance criteria, has been strongly linked to enhanced student learning (Black and Wiliam, 1998a; Black and Wiliam, 1998b; Gardner, 2006). In drawing attention to the potential of utilising assessments in support of learning, we are distinguishing assessment from the more narrowly defined goals of selection and certification typical of high-stakes examination systems. The realignment of the relationship between learning and assessment is important in considering the redesign of examination and assessment systems. In that redesigning, two aspects of assessment and examinations must be addressed: (i) *how* examinations and assessments are organised and (ii) *what* they examine/assess. Regardless of how sophisticated the design and enactment of any examination/assessment system is, it will inevitably be, in Shulman's (1987) terms, a 'union of insufficiencies' in that our visions and practices of learning more often than not outstrip our mechanisms for its assessment.

How do we assess?

Firstly we address the ways in which examinations and assessments are undertaken. The question of new assessment and examination traditions arises, given the importance of the latter in enshrining as almost sacred particular forms of knowledge. One of the advantages of an 'exam tradition' in any educational culture is the very fact that it is reflected in some degree of shared understanding about what knowledge is valued, how all involved (students, families and teachers as well as examinations administrators, test/question designers, curriculum developers, media commentators) can prepare for the high-stakes examinations, how students go about the actual exam (typically, a sit-down paper-and-pencil mode of assessment), and most importantly there is typically a very significant degree of credibility attached to the results in terms of both their validity and fairness. Madaus (1992) catalogues the advantages of highstakes examinations as follows (p. 229):

- they are relatively objective and impartial means of distributing educational benefits
- they engender a degree of national homogeneity in educational standards and practice
- they give teachers a sense of purpose and provide tangible benefits for students
- they diminish the conflict between roles of teaching and assessment by providing an assessment procedure that is unaffected by personal relationships between teachers and students
- they are widely accepted by society
- they create some sense of social and educational standards among the young, while meeting some definition of comprehensiveness, equal access, and shared experience.

These examination advantages tend to create a powerful force for stability. In the present context, given the *push* and *pull* factors which are presenting schools and society with new visions of both subject matter and learning, some examination/assessment systems are being reformed by changing assessment modes congruent with these new visions of learning (e.g. the introduction of some school-based project work and ICT-supported testing in Victoria, Australia). Furthermore, taking the learning principles underpinning RME and situated cognition-inspired PISA mathematics literacy to their logical conclusion, suggests that assessments ought to involve students in actual real-world settings with real-world problems. As Cosgrove *et al.* (2005) observe, 'Students would work in groups, each one contributing ideas on how a problem might be solved, and what the results mean' (p. 5). They then note the constraints under which PISA operates which resulted in PISA relying on 'traditional' paper-and-pencil modes of assessment. This 'practical' fall-back option is a familiar response of education systems given the logistical, bureaucratic and financial burden of designing more authentic assessments (e.g. project work, portfolios, collaborative assessment tasks) on a large scale (Mehrens, 1992). However, in the long term, the 'opportunity cost' of not providing some form of more authentic or real-world assessment modes in post-primary mathematics seems significant and might be measured in terms of student alienation and disenchantment with the subject, the persistence of a narrow vision of what counts as mathematical knowledge and competence, and decreasing the likelihood that students will develop their capabilities to use mathematical knowledge in the context of new and non-routine real-world contexts. Thus the negative systemic effects of high-stakes examinations may be added to other, well documented drawbacks of the 'traditional' timed paper-and-pencil examination format. Among the disadvantages Madaus (1992) catalogues are the following (p. 30):

- High-stakes tests in upper grades can have an undesirable backwash or trickle-down effect on class work and on study in the lower grades.

- They tend to encourage undue attention to material that is covered in the examinations, thereby excluding from teaching and learning many worthwhile educational objectives and experiences.

- Scores on them come to be regarded by parents and students as the main, if not sole, objective of education.

- They are usually carried out under artificial conditions in a very limited time frame. They are not suitable for all students and can be extremely stressful for some. In addition, they can negatively affect such personality characteristics as self-esteem and self-concept.

- There is often a lack of congruence between course objectives and examination procedures (e.g. there may be no examinations for oral or practical objectives).

- Some kinds of teaching to the test enable students to perform well on examinations without engaging in higher levels of cognition.

- Preparation for high-stakes tests often overemphasises rote memorisation and cramming by students and drill and practice as a teaching method.

- High-stakes tests are inevitably limited in the range of characteristics that they can assess, relying heavily on verbal and logico-mathematical areas.

This catalogue of disadvantages may sound all too familiar in the Irish context, but it can be seen as a source of inspiration in designing more authentic assessments in mathematics. It is also a warning about how the high-stakes nature of any type of examination or assessment can be distorted by wider socio-political and economic forces. However, as we noted in chapter three, an Australian study of the impact of alternative mathematics assessment in the state-level Victorian Certificate of Education (VCE) demonstrated that there was a positive backwash effect on teaching in Years 7-10 after changes were made in assessment practices in Years 11 and 12. Year 11 and 12 assessments consist of four components: (a) a multiple-choice skills test, (b) an extended answer analytic test, (c) a 10-hour 'Challenging Problem', and (d) a 20-hour 'Investigative Project' (Clarke and Stephens, 1996; Verschaffel, Greer and de Corte, 2000). However, as we noted earlier (see chapter three) the VCE has been revised and it now consists of only two rather than four components; that is, an examination and a school-based assessment (Barnes, Clarke and Stephens, 2000).

What is assessed?

Whether in APEC or the OECD countries, the question of what needs to be assessed has presented a new challenge in terms of the appraisal of students' 'learning to learn' capacity. One of the features of globalisation, and its attendant calls for the development of a knowledge society, is a recognition that students cannot and will not be able to learn all they need to know in school (OECD, 2003). This has pressed education systems to focus, not just on subject matter, but on the promotion of learning to learn both within subject matter and as a cross-curricular competence (Baumert, 1999). In the case of the OECD/PISA, the assessment of problem-solving as a domain (in addition to reading literacy, mathematical literacy and scientific literacy) is indicative of this new focus on not just teaching problem-solving but assessing it in a formal manner. However, assessing problem-solving separately (as a proxy indicator for learning to learn competence) seems contradictory, given Resnick's (1987) observations that learning to think/learn as well as the teaching of thinking/learning skills has to occur within some subject/content area. The actual promotion of students' learning to learn competence crystallises the relationship between teaching/learning and assessment, and demands a greater focus on a number of different types of assessment, such as assessment for learning (AfL) as well as peer and self-assessment (see chapter three). In the context of the current review of post-primary mathematics education, reforming the examination system is likely to present the biggest challenge, given the stability-inducing effect of an examination tradition, as well the logistical, bureaucratic and financial implications of reforming mathematics education examinations system-wide in line with new visions of both *learning mathematics* and *learning to learn in the context of mathematics*. However, we note that examination system reforms can have a strong ripple effect on teaching, learning and assessment down through the post-primary system, as has been the case in Victoria, Australia over the last fifteen years (Barnes, Clark and Stephens, 2000). A change in the assessment system at post-primary level may be essential in overcoming what Elwood and Carlisle (2003) see as the 'narrow view of achievement in mathematics… promoted by these examinations, and it is one that does not sit comfortably with the aims and objectives outlined in the syllabi' (p. 111). Close (2005) makes a similar point in his comparison between the content of questions in the Junior Certificate examination and the syllabus aims and objectives, and notes that only four of the ten aims outlined in the syllabus are actually assessed in the Junior Certificate examination.

> *The main goals of the Junior Certificate mathematics Syllabus would seem to focus more on mathematics needed for continuing education and*

> *less on the mathematics needed for life and work. The 10 sub-goals (objectives) listed in the syllabus have some similarities with objectives of the PISA framework but 6 of these sub-goals are not currently assessed in any formal way, including objectives relating to mathematics in unfamiliar contexts, creativity in mathematics, motor skills, communicating, appreciation, and history of mathematics (2005, p. 7).*

In summary despite the broad view of mathematical competence in PISA, there appears to be a narrow focus in terms of both style and content in Irish post-primary mathematics examinations.

Equity and excellence as policy and practice challenge

Balancing equity and excellence

Balancing the demands of excellence and equity is a perennial challenge for education systems, and it is often exacerbated by public and business demands for excellence rather than equity in high priority areas such as mathematics education. At a time when the traditional measures of human capital, such as years of education, have begun to shift towards more teaching/learning focused measures, such as those described by the reading, mathematical and scientific literacies of OECD-PISA (OECD, 2003), the relationship between excellence and equity in the distribution of human capital in these priority educational areas is an important educational and public policy issue. As we noted in chapter one, one of the reasons for international interest in Japan's post-primary mathematics education was not only its high aggregate scores but also the low variation between high and low-scoring students in the 1995 Third International Mathematics and Science Study (TIMSS). In a nutshell, Japanese post-primary mathematics education managed to achieve a greater degree of excellence and equity than other countries. In the context of PISA 2003, the Irish report (Cosgrove *et al.*, 2005) noted that the distribution of mathematical literacy scores for Irish 15-year-old students indicated that, compared to most other countries, there was a relatively low number of students scoring at the high end and a high number of students scoring at the lower end. This phenomenon presents a policy challenge: to ensure both excellence and equity within the education system. As Cosgrove *et al*. note:

> *The relatively low performance of higher-achieving students in mathematics in Ireland is noteworthy and suggests that any forthcoming review of mathematics education at post-primary education should consider this finding, with a view to identifying ways in which performance of high achievers can be enriched (p. xxiii)*

We might add here that any discussion of high achievers ought also to consider the wider public and educational policy implications of efforts to reform mathematics education in terms of its role in the distribution of mathematical literacy-based human capital.

Three capitals and mathematics education: human, social and identity capitals

The benefits of learning for people's life-chances have become a central policy focus of governments, as evidenced by the commitment to and interest in OCED/PISA, now that learning is becoming the new labour in the 21st century. High achievement in high priority areas such as mathematics is likely to contribute to significant enhancement in both individuals' and society's *human* capital (the knowledge and skills possessed by individuals). That has been the basis for massive investments in education since the 1960s (Husen, *et al*, 1992). Over the last decade, two other types of capital, social (networks of power and privilege which allow people to contribute to common goals) and *identity* (personal resources to define themselves and have others define them in a changing world), have come to the forefront in understanding the wider benefits of learning (Schuller, Preston, Hammond, Brasset-Grundy and Bynner, 2004). Noting the intertwined nature of the three capitals, Schuller *et al.* (2004) make a case for the interpretation of learning experiences and outcomes in terms of all three types of capital.

Reviewing post-primary mathematics education provides an opportunity to consider the wider benefits of mathematics education, incorporating not only human capital but also social and identity forms of capital. Inclusion of these other capitals in considering the wider benefits of mathematics might help address questions of how the access to power and privilege is related to achievement or lack of achievement in mathematics (social capital), and the role of mathematics in defining a sense of identity and personal capacity, given the manner in which numeracy is often popularly misused as a proxy measure of intelligence (identity capital), which in turn constrains or affords certain types of self-definition/classification and definition/ classification by others.

The teacher education challenge

> *…Consistent and increasing pressure on teachers, school leaders, administrators, policy-makers and researchers to construct new understandings, insights and practices to bring about transformations in the schools as organizations while simultaneously inventing more appropriate, efficient and effective approaches to teaching and learning consonant with individual needs, national aspirations and economic competitiveness . .[puts] increasing pressure on teachers to be accountable not only for the attainment and achievement of their students but also for the ways in*

which they teach . . the central message internationally is . . that business as usual for schools and teachers is no longer an adequate response to the rapidly changing landscape.

(Sugrue and Day, 2002, p. xv).

Sugrue and Day highlight the new context of teachers' work in an era of demands for higher standards and increased educational accountability. Calls for new teaching and learning approaches put pressure on teacher education at all levels in terms of how it can provide the type of experiences that will make it possible (Conway and Clark, 2003). Despite the calls for teachers to teach in new ways, initial and induction phases of teacher education and ongoing professional development opportunities often fail to live up to what is now a substantial knowledge-base concerning high-quality professional development. As Hiebert *et al.* (2002) note, 'There is a growing consensus that professional development yields the best results when it is long-term, school-based, collaborative, focused on students' learning, and linked to curricula' (p. 3). The high-quality professional development we have outlined in looking at lesson study (chapter two) and coaching of beginning, mid-career and veteran mathematics teachers (chapter four), are good examples of the type of continuing professional development necessary to begin any process of mathematics education reform. Hiebert *et al.* identify five distinguishing characteristics of quality professional development:

- elaborating the problem and developing a shared language for describing the problem
- analysing classroom practice in light of the problem
- envisioning alternatives, or hypothesising solutions to the problem

- testing alternatives in the classroom, and reflecting on their effects
- recording what is learned in a way that is shareable with other practitioners.

The criteria for high-quality teacher learning outlined by Hiebert *et al*. are consistent with a situated cognition perspective on teacher learning. Over the last fifty years, each of the three influential theories on learning - that is, the behaviourist, cognitive and sociocultural theories - have influenced conceptions of good teaching, visions of professional development and the evaluation of teachers and teaching (Conway and Artiles, 2005). As Putnam and Borko (2000) note, an influential and developing body of knowledge on cognition and learning suggests that cognition is situated, social and distributed. Thus, in terms of teacher development, teacher thinking cannot be isolated from the context of teaching, that is, the classroom and the school. Just as the situated cognition perspective reframes student learning in significant ways, so it does in the case of teacher professional development. A situated cognition view of teacher learning suggests the need to

- ground staff-development in teachers' learning experiences in their own practice by conducting it on-site at schools and in the classroom
- encourage teachers to bring experiences from their own classroom to staff development workshops that are extended over a number of weeks or months
- incorporate multiple contexts for teacher learning (both site-based, drawing on teachers' own practice, and those involving the perspectives of 'outsiders' such as in-service providers, inspectors, university lecturers, etc.).

In contrast to the high quality professional development outlined by Hiebert *et al* and supported by insights from situated cognition perspective on teacher learning, teacher professional development typically falls far short of these standards.

As Fullan has noted:

> *Professional development for teachers has a poor track record because it lacks a theoretical base and coherent focus. On the one hand, professional development is treated as a vague panacea - the teacher as continuous, lifelong learner. Stated as such, it has little practical meaning. On the other hand, professional development is defined too narrowly and becomes artificially detached from 'real-time' learning. It becomes the workshop, or possibly the ongoing series of professional development sessions. In either case, it fails to have a sustained cumulative impact. (Fullan, 1995, cited in Guskey and Huberman, 1995, p. 253)*

In light of Huberman's observations we draw on two studies to highlight key issues in relation to teacher professional development in the Irish context ([i] Sugrue, Morgan, Devine, and Raftery, 2001; Sugrue, 2002; and [ii] Delaney, 2005). Firstly Sugrue, Morgran, Devine and Raftery's (2001) Department of Education and Science-commissioned survey and interview study of primary and post-primary teachers about their professional learning provides a good overview of the state of teacher professional development. We highlight number of their main findings here. Key claims of relevance to this report are as follows:

- Teachers were generally positive about their continuing professional development (CPD) experiences.

- Teachers said they felt their CPD was making some impact on classroom practice. Sugrue (2002), in subsequent article drawing on the same data, notes that in the absence of evidence linking teachers' professional learning opportunities with their classroom practices and ultimately student learning, many questions about the impact of CPD remain unanswered.

- Teachers felt that 'professional learning provision has been more successful in communicating cognitive knowledge than impacting positively on competencies and skills' (p. 334).

- The approach to provision of CPD tended to favour 'talking at' teachers rather than more interactive approach where teachers have a chance to explore ideas with others. Sugrue (2002) notes that even when CPD was more interactive, there was little support for teachers once they returned to their schools in terms of addressing issues raised at professional learning sessions.

Thus, this first study portrays a professional learning landscape that is at one level satisfactory (teachers are generally positive about it), but at another appears to leave teachers poorly served in terms of the kind of professional learning needed to initiate and sustain significant changes in teachers' classroom practices.

Delaney (2005), in a study reviewing professional development in mathematics for primary teachers, noted that, based on the 1999 National Assessment of Mathematics Achievement, only just over a quarter of teachers (29%) had attended in-service training in mathematics, and of these just less than half (47%) were either dissatisfied or very dissatisfied with the courses they attended. Furthermore, he notes that the one-day and one-week professional development opportunities are unlikely to provide the time needed to develop a discourse with which to share and examine mathematics teaching practices in a collegial setting. In summary, any radical reform of mathematics education in Ireland presents a challenge to create the continuing professional development CPD structures and opportunities which are needed to support it.

Scaling up:[22] the change challenge

> *'…An ecological model of education goes far beyond schools in seeking to embrace people, things and institutions in a systemic, interrelated whole.'*
>
> **(Goodlad, 1975)**

Given the acceleration in social, economic and cultural change around the world, education systems have been asked to (and asked themselves to) meet the change challenge as it relates to almost every aspect of the educational environment within their control. This has placed both a heavy burden and an exciting range of possibilities at the feet of educators and all those involved in education. Mathematics education has been made a policy priority, and presents an especially urgent challenge in curriculum and assessment. Educational change researchers have not been very optimistic about the capacity of educational systems to change, noting the powerful forces for stability often very useful in preserving what is valuable in a culture and education system. The title of Sarason's classic text *The Predictable Failure of Educational Reform*, based on analyses of change initiatives in the USA over several decades, captures the insights of those sceptical of revolutionary change agendas in education. In Ireland today, one important source of support for change is the individual school's own capacity to change as a result of the rolling

22 See Coburn (2003) for an analysis of how scale has been conceptualised in the educational change literature. She notes that 'definitions of scale have traditionally restricted its scope, focusing on the expanding number of schools reached by a reform' (p. 3). Coburn argues for multidimensional conception of scale that addresses its interrelated dimensions: depth, sustainability, spread, and shift in ownership.

review of policies and practices brought into motion by School Development Planning (SDP) and Whole School Evaluation (WSE) processes.

However, reforming mathematics education will need to go beyond school level capacity for change, and adopt a system-level approach given the scale of the task involved. The National Council for Curriculum and Assessment proposals for senior cycle sets out possible reform framework for upper post-primary education (NCCA, 2003; NCCA 2005b). These wider reforms outlined by the NCCA set an essential context for any proposals to reform post-primary mathematics education. Developing a different school culture, reforming students' experiences of teaching and learning, re-balancing curriculum at senior cycle, and developing different assessment arrangements and a new certification system for senior cycle, were the directions for development specified in the NCCA document. These documents outlined a series of support strategies that would be necessary to achieve these ambitious goals, encompassing investment for change, professional development for teachers and support for schools, provision of information to and engagement with stakeholders and parents, and monitoring, research and evaluation. What might be the best place to start in terms of reform? Assuming that some significant reform agenda is adopted, the post-primary mathematics education in Victoria, Australia reform strategy[23] provides some evidence that reforming the assessment system can produce a significant ripple effect right through secondary education. It is examples such as the VCE reforms and the insights on the mismatch between mathematics textbooks and mathematics education reform goals that identify potential system-level levers for change.

[23] See VCAA, 2004 and 2005 for relevant resources.

5.4 Conclusion

Mathematics education has become the focus of considerable attention in Ireland and elsewhere since mathematical competence is seen as playing a critical role in the development of a knowledge society, as well as an essential skill for productive and reflective citizenship. Recent research on post-primary mathematics education in Ireland has begun to question the dominance of the 'new' mathematics education movement and its impact on approaches to teaching and examinations. In this report we have not attempted to outline a strategy for reforming mathematics education but have focused on key aspects that might inform such a strategy, such as the role of curriculum culture, textbooks and examinations. From our review of the trends, it is clear that there is no one template for reforming post-primary mathematics education. There are, however, trends, of which the move towards a more 'real-life' focus in mathematics curriculum and assessment is the most distinctive and significant shift in mathematics education in many countries.

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