UNIVERSITY OF MANCHESTER

Leaving Certificate Mathematics: a comparative analysis

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Leaving Certificate Mathematics: a comparative analysis

This report compares the Leaving Certificate Mathematics syllabus for examination from 2015 with five other countries/ jurisdictions.

Contents

Introduction

This work was commissioned by Ireland's National Council for Curriculum and Assessment (NCCA) in Spring 2013, with a view to seeing how Ireland's Leaving Certificate Mathematics syllabus under the *Project Maths* initiative compares with curriculum expectations around the world.

The following countries/jurisdictions were used in the comparison: Finland, Massachusetts, New Zealand, Singapore and Scotland. They represent a subset of the countries analysed by Hodgen et al. (2013) in the Nuffield study: *Towards universal participation in post-16 mathematics: lessons from high-performing countries* which looked in depth at countries with relatively high participation rates identified in their 2010 study: *Is the UK an outlier?* (Hodgen et al. 2010a) These countries are most similar to Ireland in terms of participation in upper secondary mathematics. The table below shows how the selected countries compare with Ireland[1].

Extract from table 6: What are the participation rates in upper secondary mathematics education? (Hodgen et al. 2010a: 38)

Country	**Not studying mathematics**	**Studying any mathematics**	**Studying advanced mathematics**	**Compulsory/ proportion of curriculum**
Ireland (NCCA data)	Negligible: 0-5%	All: 95-100%	Low: 0-15%	N, one of seven
Finland	Negligible: 0-5%	All: 95-100%	Medium: 16-30%	Y, one of seven or eight
Massachusetts (USA)	Few: 6-20%	Most: 81-94%	Medium: 16-30%	Y, one of seven
New Zealand	Some: 21-50%	Many: 51-80%	High: 31-100%	Y, one of eight
Scotland	Many: 51-80%	Some: 21-50%	Medium: 16-30%	N, one of five
Singapore	Some: 21-50%	Many: 51-80%	High: 31-100%	N, one of six

Three of these six countries include mathematics as a compulsory subject in the upper secondary curriculum. Of those that don't include mathematics as a compulsory subject in the upper secondary curriculum, two of them (Scotland and Singapore) have a narrower curriculum than the other countries.

This report briefly sets out the provision in each country under the following themes:

- Curriculum Aims/Objectives
- Proportion of curriculum
- Strands and areas of study
  - depth of treatment
  - progression across levels
- Evidence of problem-solving approach
- Assessment of syllabus objectives

[1] Note that the data for Ireland has been provided by NCCA.

These are then compared with Ireland's Leaving Certificate Mathematics. The final section of this report summarises the findings from the comparisons and includes:

- Observations of what is included/not included in Ireland's Leaving Certificate Mathematics
- The extent to which connections between topics are made
- Problem-solving approaches
- Differentiation and progression issues
- Clarity of the strands of study and the learning outcomes

Only the publicly available documentation concerned with the curriculum and assessment for upper secondary level were scrutinised for this report.

The countries/jurisdictions

Ireland

The Leaving Certificate Mathematics curriculum aims to develop knowledge, skills and understanding for study, life and work. The objectives for Leaving Certificate Mathematics are:

- the ability to recall relevant mathematical facts
- instrumental understanding ("knowing how") and necessary psychomotor skills (skills of physical co-ordination)
- relational understanding ("knowing why")
- the ability to apply mathematical knowledge and skill to solve problems in familiar and unfamiliar contexts
- analytical and creative powers in mathematics
- an appreciation of mathematics and its uses
- a positive disposition towards mathematics (NCCA, 2012: 2)

Students study mathematics alongside six other subjects. The vast majority study mathematics as it is essential for entry to higher education.

The content is set out in five strands: Statistics and Probability; Geometry and Trigonometry; Number; Algebra and Functions, at three different levels: Foundation, Ordinary and Higher. The levels are progressive and a student working at ordinary level is expected to be competent at foundation level material. At higher level students are expected to have engaged with learning outcomes at both foundation and ordinary level. The content for each level is helpfully set out alongside one another, which is unique to Ireland at this level of study amongst the countries investigated.

The development of synthesis and problem-solving skills is articulated through each content strand of the curriculum:

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions (NCCA 2012: 19)

At each of the three levels students are assessed by two examination papers, for which calculators are allowed, comprising two sections: section A: core mathematics topics with a focus on concepts and skills, section B: context-based applications (NCCA, 2012: 85).

In addition there is an Applied Mathematics qualification that is taken by very few students and assesses material beyond the Leaving Certificate Mathematics syllabus. As it is currently under review it has not been included in this comparison.

Finland

The upper secondary curriculum for mathematics has two pathways: Advanced and Basic, set out in just ten pages. They have the following objectives:

Objectives of Basic syllabus	Objectives of Advanced syllabus
• be able to use mathematics as an aid in everyday life and social activities; • obtain positive learning experiences when working with mathematics and learn to trust their own abilities, skills and thinking; find courage to engage in experimental, exploratory and inventive learning; • acquire such mathematical skills, knowledge and capabilities that will create a sufficient foundation for further studies; • internalise the significance of mathematics as a tool which can be used to describe, explain and model phenomena and to draw conclusions; • form an overview of the nature of mathematical knowledge and its logical structure; • gain practice in receiving and analysing information provided by the media in a mathematical form and in assessing its reliability; • acquaint themselves with the significance of mathematics in the development of culture; • learn to use figures, formulae and models in support of thinking. (Finnish National Board of Education, 2003: 129)	• become accustomed to persistent work, thus learning to trust their own mathematical abilities, skills and thinking; • find courage to adopt experimental and exploratory approaches, discover solutions and assess these critically; • understand and be able to use mathematical language, so as to be capable of following mathematical presentations, reading mathematical texts and discussing mathematics, and learn to appreciate precision of presentation and clarity of argumentation; • learn to perceive mathematical knowledge as a logical system; • develop their skills to process expressions, draw conclusions and solve problems; • gain practice in processing information in a way characteristic of mathematics, become accustomed to making assumptions, examining their validity, justifying their reasoning and assessing the validity of their arguments and the generalisability of the results; • gain practice in modelling practical problem situations and making use of various problem-solving strategies; • know how to use appropriate mathematical methods, technical aids and information sources. (Finnish National Board of Education, 2003: 123)

All students have to study mathematics as part of a curriculum which includes six or seven other subjects. The basic course comprises six compulsory courses and two specialisation courses. The advanced mathematics course comprises ten compulsory courses and three specialisation courses. It is possible for students to move from the advanced course to the basic course (see the mapping of core courses in the summary table below).

Advanced courses (core)	Basic Courses (core, includes mapping)
MAA1 Functions and Equations	MAB1 Expressions and Equations
MAA2 Polynomial Functions	
MAA3 Geometry	MAB2 Geometry
MAA4 Analytical Geometry	
MAA5 Vectors	
MAA6 Probability and Statistics	MAB5 Statistics and Probability
MAA7 The derivative	MAB4 Mathematical Analysis
MAA8 Radical and logarithmic functions	MAB3 Mathematical models I
MAA9 Trigonometric functions and number sequences	
MAA10 Integral calculus	
	MAB6 Mathematical models II
Advanced courses (specialisation)	**Basic Courses (Specialisation)**
MAA11 Number theory and logic	MAB7 Commercial mathematics
MAA12 Numerical and algebraic methods	MAB8 Mathematical models 3
MAA13 Advanced differential and integral calculus	

As part of the Finnish Matriculation examination, mathematics is assessed at either the Basic or Advanced level through a 15 item, six hour test based on all the compulsory and specialisation courses. Students are expected to complete ten questions and may use calculators and formulae booklets.

Massachusetts (USA)

The Massachusetts curriculum framework for mathematics was revised in 2011. It defines eight standards for mathematical practice that describe mathematical proficiency:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning (Massachusetts Curriculum Framework for Mathematics, 2011: 15-17)

All students have to study mathematics as one of seven subjects. For high school mathematics (grades 9 to 12, ages 15 to 18) the mathematics standards are presented by conceptual category:

- Number and quantity
- Algebra
- Functions
- Modelling
- Geometry
- Statistics and probability (Massachusetts Curriculum Framework for Mathematics, 2011: 66)

Except for modelling, the conceptual categories are further subdivided into domains with a unique code under which there will be a number of individual standards. The modelling cycle is included and aspects of the other standards relevant to modelling are starred. The table below summarises the domains and number of associated standards (summary of Massachusetts Curriculum Framework for Mathematics, 2011: 67-94).

Conceptual Category	Domain	Code	Standards
Number and Quantity	The real number system	N.RN	3
	Quantities*	N.Q	3
	The complex number system	N.CN	9
	Vector and matrix quantities	N.VM	12
Algebra	Seeing structure in expressions	A.SSE	4
	Arithmetic with polynomials and rational expressions	A.APR	7
	Creating equations*	A.CED	4
	Reasoning with equations and inequalities	A.REI	12
Functions	Interpreting functions*	F.IF	10
	Building functions*	F.BF	5
	Linear, quadratic and exponential models*	F.LE	5
	Trigonometric functions	F.TF	9
Geometry	Congruence	G.CO	13
	Similarity, right triangles and trigonometry	G.SRT	11
	Circles	G.C	5
	Expressing geometric properties with equations	G.GPE	7
	Geometric measurement and dimension	G.GMD	4
	Modelling with geometry*	G.MG	4
Statistics and Probability	Interpreting categorical and quantitative data*	S.ID	9
	Making inferences and justifying conclusions*	S.IC	6
	Conditional probability and the rules of probability*	S.CP	9
	Using probability to make decisions*	S.MD	7

These standards are arranged into model courses: traditional (Algebra I, Geometry, Algebra II) and integrated (Mathematics I, II, III). In addition there are two advanced model courses: Pre-calculus and Quantitative methods.

The table below shows the intended progression and coherence by mapping the conceptual categories, domains and standards to the integrated mathematics and advanced courses (summary of Massachusetts Curriculum Framework for Mathematics, 2011: 119-152).

	Number and quantity	**Algebra**	**Functions**	**Geometry**	**Statistics and probability**
Mathematics I	N.Q 1-3	A.SSE 1 A.CED 1-4 A.REI 1,3,5, 10-12	F.IF 1-10 F.BF 1-3 F.LE 1-5	G.CO 1-8,12 G.GPE 4,5,7	S.ID 1-3,5-9
Mathematics II	N.RN 1-3 N.CN 1,2,7-9	A.SSE 1-3 A.APR 1 A.CED 1,2,4 A.REI 4,7	F.IF 4-8,10 F.BF 1,3,4 F.LE 3 F.TF 8	G.CO 9-11 G.SRT 1-8 G.C 1-5 G.GPE 1-4,6 G.GMD 1,3	S.CP 1-9 S.MD 6,7
Mathematics III	N.CN 8,9	A.SSE 1,2,4 A.APR 1-7 A.CED 1-4 A.REI 2,11	F.IF 4-9 F.BF 1,3,4 F.LE 4		S.IC 1-6 S.MD 6,7
Pre-calculus	N.CN 3-6,8,9 N.VM 1-12	A.APR 5-7 A.REI 8,9	F.IF 7 F.BF 1,4,5 F.TF 3,4,6,7,9	G.SRT 9-11 G.C 4 G.GPE 3 G.GMD 2,4	
Quantitative reasoning	N.VM 1-12	A.APR 5 A.REI 8,9	F.TF 3,4,5,7,9	G.SRT 11 G.C 4 G.GPE 3 G.GMD 2,4 G.MG 3,4	S.ID 9 S.IC 4-6 S.CP 8,9 S.MD 1-7

The Massachusetts Comprehensive Assessment System (MCAS) assesses high school mathematics in grade 10. MCAS requires that Number & quantity, Algebra & functions, Geometry and Statistics & probability are weighted 2:3:3:2. The vast majority of students take MCAS, which comprises external tests in mathematics, English language arts (ELA) and one of four science or engineering technology. MCAS grade 10 is a component of the high school diploma and is the basis of school accountability. The MCAS grade 10 test comprises two one hour papers: one without a calculator comprising multiple choice and short response items and one with a calculator comprising short response and open response items. In order to enter higher education, students must successfully complete the high school courses Mathematics I, II and III or the traditional alternative.

New Zealand

The New Zealand curriculum was revised in 2007 following a comprehensive review of international research (Anthony & Walsh, 2007). Mathematics and statistics is one of eight learning areas for years 1 to 13 of compulsory schooling (age 5 to 18). The objectives are:

1. Inspire thinking

Mathematics and statistics help make sense of information, experience, and ideas by engaging students to think:

- flexibly and creatively
- critically and effectively
- strategically and logically

2. Stimulate creativity and curiosity

Mathematics and statistics open the door to a world of beauty, mystery, and awe. They provide students with the enjoyment of intellectual challenge: opportunities to explore ideas and to wrestle with interesting problems. Mathematics and statistics provide ways of connecting abstract ideas with real world thinking.

3. Equip students for the 21st century

Mathematics and statistics equip students with the knowledge and skills to be global citizens in the 21st century. Effective citizens have the ability and inclination to use mathematics and statistics at home, at work, and in the community by:

- using problem-solving strategies
- using mathematical and statistical models to solve problems
- making sensible estimates
- using and interpreting data
- evaluating mathematical and statistical information
- communicating ideas (http://seniorsecondary.tki.org.nz/Mathematics-and-statistics/Rationale)

Content requirements are set out as achievement objectives in eight levels. Levels 6, 7 and 8 comprise the upper secondary expectations. At level 6 there are three strands for mathematics and statistics: Number and algebra (NA), Geometry and measurement (GM) and Statistics (S), and just two strands at levels 7 and 8: Mathematics (M) and Statistics (S).

Mathematics and statistics is assessed as part of the National Certificate of Educational Achievement (NCEA) available at levels 1 to 3 (roughly equivalent to achievement objectives levels 6, 7 and 8 respectively). The NCEA comprises short coherent units at the three levels which are largely internally assessed. At level one there is an externally set one hour, non-calculator mathematics common assessment task (MCAT) taken by most students in year 11. The other level one units comprise nine internally assessed units and three externally assessed units. At level two there are three externally assessed units and eleven internally assessed units, and at level three of the 15 available units, six are externally assessed. The units allow students to build up a personal profile of mathematical achievement that contributes to their overall upper secondary education achievement. Apart from MCAT, externally assessed units are sat three at a time in a three hour sitting. MCAT is unusual in that it is internally marked and the marking is externally moderated. Internally assessed units are subject to both internal and external moderation.

NCEA assessment units for mathematics and statistics (assessed internally and **externally**) (summarised from http://ncea.tki.org.nz/Resources-for-aligned-standards/Mathematics-and-statistics)

Level 1	Level 2	Level 3
Apply numeric reasoning in solving problems	Apply co-ordinate geometry methods in solving problems	Apply the geometry of conic sections in solving problems
Apply algebraic procedures in solving problems (MCAT)	Apply graphical methods in solving problems	Apply linear programming methods in solving problems
Investigate relationships between tables, equations and graphs	Apply sequences and series in solving problems	
Apply linear algebra in solving problems	Apply trigonometric relationships in solving problems	Apply trigonometric methods in solving problems
Apply measurement in solving problems	Apply network methods in solving problems	Use critical path analysis in solving problems
Apply geometric reasoning in solving problems	**Apply algebraic methods in solving problems**	**Apply the algebra of complex numbers in solving problems**
Apply right-angled triangles in solving measurement problems	**Apply calculus methods in solving problems**	**Apply differentiation methods in solving problems**
		Apply integration methods in solving problems
Apply knowledge of geometric representations in solving problems	Design a questionnaire	Investigate time series data
		Investigate bivariate measurement data
Apply transformation geometry in solving problems	Use statistical methods to make an inference	Use statistical methods to make a formal inference
Investigate a given multivariate data set using the statistical enquiry cycle	Conduct an experiment to investigate a situation using statistical methods	Conduct an experiment to investigate a situation using experimental design principles
Investigate bivariate numerical data using the statistical enquiry cycle	Evaluate a statistically based report	**Evaluate statistically based reports**
Demonstrate understanding of chance and data	**Apply probability methods in solving problems**	**Apply probability concepts in solving problems**
Investigate a situation involving elements of chance	Investigate a situation involving elements of chance using a simulation	**Apply probability distributions in solving problems**
	Apply systems of equations in solving problems	Apply systems of simultaneous equations in solving problems

Scotland

The curriculum in Scotland has undergone considerable revision with the recent introduction of the Curriculum for Excellence. The overarching aims are that each young person becomes a successful learner, a confident individual, a responsible citizen and an effective contributor. Numeracy and mathematics is one of eight subject areas, set out in terms of experiences and outcomes at early, first, second, third and fourth stages (typically age 16). The curriculum is written in learner-friendly language including the objectives: My learning in mathematics enables me to:

- develop a secure understanding of the concepts, principles and processes of mathematics and apply these in different contexts, including the world of work
- engage with more abstract mathematical concepts and develop important new kinds of thinking
- understand the application of mathematics, its impact on our society past and present, and its potential for the future
- develop essential numeracy skills which will allow me to participate fully in society
- establish firm foundations for further specialist learning
- understand that successful independent living requires financial awareness, effective money management, using schedules and other related skills
- interpret numerical information appropriately and use it to draw conclusions, assess risk, and make reasoned evaluations and informed decisions
- apply skills and understanding creatively and logically to solve problems, within a variety of contexts
- appreciate how the imaginative and effective use of technologies can enhance the development of skills and concepts (Numeracy and Mathematics experiences and outcomes, available at http://www.educationscotland.gov.uk/learningteachingandassessment/curriculumareas/mathematics/eandos/index.asp)

Upper secondary students usually study five subjects. Although mathematics is not compulsory it is a popular choice, with just under half of all students studying some mathematics in upper secondary and just over half of these studying advanced mathematics (Higher). A much smaller proportion study the Advanced Higher which goes beyond the expectations of most other countries in this comparison. Students studying the Advanced Higher typically do so with just two other subjects and may progress directly to the second year of the four-year degree courses that are typical of Scottish universities (Hodgen et al, 2013)

The table below summarises the content strands in the Scottish curriculum (http://www.educationscotland.gov.uk and http://www.sqa.org.uk):

Early – Fourth Stage	**Intermediate**	**Higher**	**Advanced Higher**
Number, money and measure Shape, position and locations Information handling	Arithmetic Algebra Geometry Trigonometry Statistics	Algebra Geometry Trigonometry Statistics Elementary calculus	Algebra Geometry Calculus

All of the mathematics qualifications (http://www.sqa.org.uk) have the same problem solving objectives:

- Interpret the problem and consider what might be relevant
- Decide how to proceed by selecting an appropriate strategy
- Implement the strategy through applying mathematical knowledge and understanding, and come to a conclusion
- Decide on the most appropriate way of communicating the solution to the problem in an intelligible form

At age 16 most students take the Standard Grade (SG) qualification which is available at three levels: Foundation, General and Credit. For the fifth stage or senior phase (upper secondary) the curriculum is set out in the qualifications: Intermediate 1 (SG general), Intermediate 2 (SG credit), Higher and Advanced Higher. There are also Access qualifications for those working below the level expected for most students at the end of the fourth stage. Each qualification comprises three units assessed internally (usually through a closed book test taken under controlled conditions, available from the National Assessment Bank) and an externally set and marked exam. The table below summarises the duration and papers.

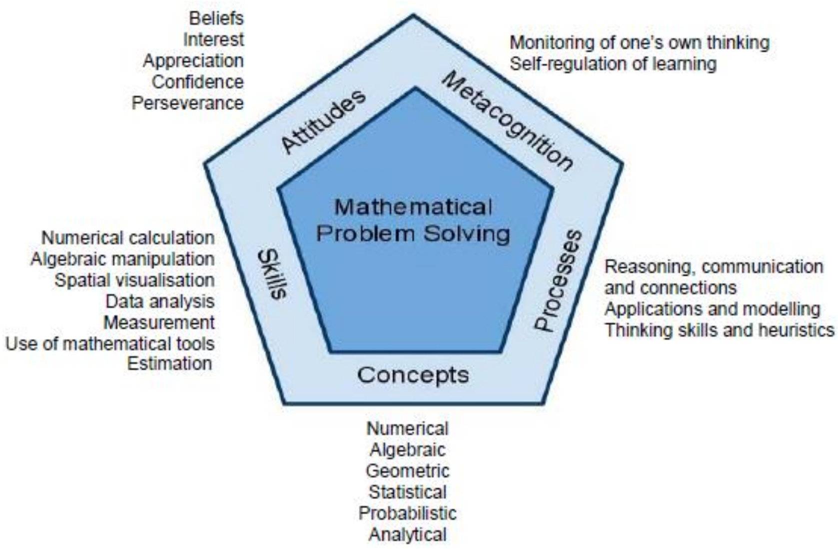
	Intermediate 1	**Intermediate 2**	**Higher**	**Advanced Higher**
Total duration	1h 30min	2h	2h 40min	3h
Papers	1. Non-calculator 2. Calculator	1. Non-calculator 2. Calculator	1. Non-calculator 2. Calculator	1. Calculator

Singapore

The curriculum in Singapore has undergone considerable change recently. The changes are being introduced year by year through the four years of the secondary phase and will be complete by 2016. The broad aims for the mathematics curriculum are:

- Acquire and apply mathematical concepts and skills
- Develop cognitive and metacognitive skills through a mathematical approach to problem solving
- Develop positive attitudes towards mathematics (Teaching and Learning syllabus, Chapter 1)

Approximately two thirds of all students study mathematics during upper secondary education and about 40% study advanced mathematics (H1 or H2). Students usually study six subjects in upper secondary and it is expected that there is breadth through a contrasting subject. Problem solving is at the heart of the Singapore curriculum and has been since 1990. The problem solving framework (below) stresses conceptual understanding, skills proficiency and mathematical processes and is used by all teachers of mathematics.



(Teaching and Learning Syllabus, Chapter 2)

The content strands in the new secondary curriculum are set out as Number and algebra; Geometry and measurement; and Statistics and probability with a Mathematical processes strand which emphasises modelling, problem-solving and making connections. (Teaching and Learning Syllabus, Chapter 3). These strands are used for O level. At A level pure mathematics and statistics are distinct strands.

	H1	H2	H3
Pure mathematics	Functions and graphs Calculus	Functions and graphs Sequences and series Vectors Complex numbers Calculus	Functions and graphs, Sequences and series, and Calculus Differential equations as mathematical models
Statistics		Permutations, combinations and	Combinatorics

	Probability Binomial and normal distributions Sampling and hypothesis testing Correlation and regression	probability Binomial, Poisson and normal distributions Sampling and hypothesis testing Correlation and regression	

Singaporean qualifications are hierarchical and summarised below (adapted from Teaching and Learning Syllabus, Chapter 1):

	Academic		Vocational
Pre-university		A level H3	
	A level H1	A level H2	
Secondary		O level Additional mathematics	
	O level		
	National (academic) level		National (technical) level

For post-16 academic students, the A level qualifications are available at H1, H2 and H3. H1 is a subset of H2 and is not taken by students who have completed O level Additional Mathematics – they can progress directly to H2. H3 is aimed at the highest attainers and develops content from H2 and extends the use of differential equations to model real phenomena including mechanics and populations. This content is beyond that of other countries.

All of the qualifications are externally assessed. In all examinations technology is expected: calculators at O level and graphing calculators at A level. The table below summarises duration and number of items.

	O level	A level H1	A level H2	A level H3
Paper 1	2h 25 short answer items – fundamental skills and concepts	3h A pure 5 items B statistics 6-8 items	3h pure 10-12 items	3h Entire syllabus About 8 items
Paper 2	2h 30mins 10-11 items - higher order thinking		3h A pure 3-4 items B statistics 6-8 items	

Comparing the countries/jurisdictions

Comparing curriculum aims and objectives

The table below summarises the curriculum aims and objectives for the six countries. Note that the aims and objectives for comparator countries have been rephrased and reordered to match those of Ireland.

Ireland	Finland	Massachusetts	New Zealand	Scotland	Singapore
factual recall					
knowing how		use appropriate tools strategically; attend to precision	carry out procedures flexibly and accurately	essential numeracy skills	skills, processes
knowing why	mathematics as a logical system	reason abstractly and quantitatively, develop understanding; look for and make use of structure and pattern	estimate with reasonableness, understand when results must be interpreted with a degree of uncertainty	secure understanding of mathematics	concepts
application to problems	mathematics for modelling and problem solving	make sense of problems and persevere in solving them	wrestle with interesting problems	apply skills and understanding creatively and logically to solve problems	mathematical problem solving
analytical and creative powers	draw conclusions, assess reliability of information; assess validity, justify reasons	stimulate curiosity, construct viable arguments and critique the reasoning of others	think creatively, critically, strategically and logically	interpret information, draw conclusions, assess risk, reasoned evaluations, informed decisions	thinking skills, metacognition
appreciation of uses	contribution of mathematics to culture and society	model with mathematics	practical applications in everyday life, other learning areas and the workplace	understand the application of mathematics now and in the past; financial capability	mathematical problem solving

positive disposition	trust their own abilities; persistence & courage	create enjoyment of mathematics	enjoy intellectual challenge	firm foundations for further specialist learning	positive attitudes towards mathematics
				use of technology	

There is considerable commonality: problem solving, procedural competence, relational understanding, positive dispositions, and analytical and critical thinking. Only Ireland has factual recall as an objective for mathematics education at this level. Only Scotland explicitly mentions technology but Ireland, Finland and Singapore expect technology use in all exams. Massachusetts and New Zealand have some external assessments without a calculator. Most countries have an objective around appreciation of mathematics but only Scotland explicitly mentions the history of mathematics and Finland mentions the contribution of mathematics to culture. Scotland also includes personal financial capability as an objective for mathematics. Although this is not an objective for other countries, many include arithmetic related to personal finance in the content for lower levels (see the content analysis).

Comparing content strands

The table below summarises the main content strands for each country. Note that the aims and objectives for comparator countries have been rephrased and reordered to match those of Ireland.

Ireland	Finland	Massachusetts	New Zealand	Scotland	Singapore
Statistics and probability	Probability and statistics	Statistics and probability	Statistics	Statistics and probability	Statistics and probability
Geometry and trigonometry	Geometry	Geometry	Geometry and measurement	Geometry	Geometry and measurement
Number	Commercial arithmetic (B) Number theory (A)	Number and quantity	Number and algebra	Arithmetic	Number and algebra
Algebra	Algebra	Algebra	level 7 & 8 Mathematics includes functions and algebra	Algebra	
Functions and calculus	Functions and calculus	Functions-*not calculus* - includes trigonometry		Algebra & trigonometry	Calculus (A level)
Synthesis and Problem solving skills	Mathematical modelling	Mathematical modelling	Problem-solving and modelling	Problem-solving	Mathematical processes

All countries include Statistics and Probability. New Zealand's Statistics incorporates Probability. Finland and Scotland have relatively little Statistics in their curricula. New Zealand has Mathematics and Statistics as the named subject area (rather than Mathematics) and, like Ireland and Massachusetts, has a greater emphasis on statistical literacy. New Zealand (level 8), Scotland (H) and

Singapore (H1) include linear regression. New Zealand and Singapore combine Number and Algebra into a single strand. Some countries like Ireland combine Geometry and Trigonometry into a single strand but others include Trigonometry as part of Functions (e.g. Scotland).

There is a detailed mapping of the different countries' content against the strands and levels of the Ireland curriculum in the appendix. As evidenced in the mapping many countries have a similar progression to Ireland. The main areas of difference are in Statistics and Geometry. Finland and Scotland have relatively little statistics and Scotland and Singapore have relatively little synthetic geometry. Finland is most different with little of its Basic syllabus mapping to Ireland's Foundation level. Finland's progression is also distinctive, for example geometric and arithmetic progressions are in the Basic syllabus whereas they are in Ireland's Higher level. Ireland is consistent with most of the other countries.

Ireland and Massachusetts have a greater emphasis on synthetic geometry at all levels compared with Finland, New Zealand, Scotland and Singapore. The latter three countries have a similar emphasis at the lower levels but not in their advanced mathematics curriculum.

Massachusetts is unique in not including calculus in its senior high school curriculum. At the advanced level, many countries go beyond Ireland in the study of calculus. Finland (advanced), New Zealand (level 8), Scotland (Advanced Higher) and Singapore (H2, H3) all include differential equations. Finland (advanced) includes volumes of integration. Singapore (H2, H3) and Scotland (AH) include MacLaurin series and Scotland (AH) includes calculus of inverse trigonometric functions.

All of the countries considered except Finland include complex numbers although most introduce them at a level beyond that of Ireland (OL), and the extent of development is less than in Ireland. For example, only Scotland (AH) and Singapore (H2) include De Moivre's theorem for a very small proportion of students.

Although Ireland (OL) includes applying differentiation to rates of change, it does not explicitly mention speed-time and distance-time unlike Scotland (Intermediate 1) and Singapore (O level). Linear programming is included in Finland (MAB6) and Massachusetts refers to using inequalities for modelling (A.CED 3). Transformation of functions is explicitly mentioned in the curriculum for Massachusetts (F.BF 3), Scotland (A, AH) and Singapore (H2). Conic sections are included in the Massachusetts curriculum (G.GPE – advanced courses only)

New Zealand (levels 7 and 8) includes network diagrams and critical path analysis. Scotland also includes networks (Intermediate 1 and 2). Finland has an advanced specialisation number theory course which includes Euclid's algorithm; Euclid's algorithm is also included in Scotland's Advanced Higher.

Unlike Ireland, Finland, Massachusetts, Scotland and Singapore include vectors, although Massachusetts only deals with vectors in advanced courses. Massachusetts (advanced courses) and Scotland (AH) include matrices and using matrices to solve systems of simultaneous linear equations. Singapore includes matrices and sets at O level.

The detailed mapping shows that there is considerable consensus amongst the six countries around the content of upper secondary mathematics. Some of the exceptions could be attributed to the breadth of the upper secondary curriculum and whether or not mathematics is compulsory. For example, Singapore (H3) and Scotland (AH) have a relatively narrow curriculum and allow greater mathematics specialisation than countries where mathematics is compulsory or a minimum of seven subjects is studied (Ireland, Finland, Massachusetts and New Zealand).

Comparing approaches to problem-solving

The table below summarises the key features of each country's approach to problem-solving. Note that Finland, Massachusetts and New Zealand place greater emphasis on modelling than on problem-solving. There is considerable consistency between each country's aspirations for their learners to be able to use their mathematics for making sense of situations and reaching justifiable conclusions.

Ireland	Finland	Massachusetts	New Zealand	Scotland	Singapore
explore patterns	exploratory approaches	try different approaches	seek patterns and generalisation		simplify a problem, consider special cases
formulate conjectures	discover solutions	develop conjectures	conjecture	interpret the problem	make inferences
justify conclusions	justify reasoning, assess validity of arguments and generalisability of results	construct viable arguments, review conclusions	justify and verify solutions and conclusions		justify solutions
communicate mathematics	use mathematical language	communicate results and respond to others' arguments	process and communicate information	communicate the solution	write mathematical arguments
solve problems in familiar and unfamiliar contexts	model practical problem situations	solve progressively deeper, broader and more sophisticated problems	wrestle with interesting problems	implement the problem-solving strategy	solve problems in a variety of contexts
translate information into mathematical form	use appropriate mathematical methods, technical aids and information sources	model with mathematics	create and use mathematical and statistical models to solve problems	interpret the problem	mathematical modelling process
devise, select and use appropriate mathematical models, formulae or techniques		try different approaches		select an appropriate strategy	use appropriate mathematics
draw relevant conclusions	draw conclusions				draw logical conclusions

Comparing approaches to assessment

There is considerable variation across the countries in how assessment is carried out. Like Ireland, Singapore is 100% external assessment. In Scotland there is unit assessment that is internally administered and assessed, alongside an externally assessed exam. New Zealand has a unitised system: most units are internally assessed and some are externally assessed through one hour exams that are usually taken in a three hour sitting. New Zealand's unitised system allows for personalisation of learning, which has increased participation in mathematics by more than 10% (Hodgen et al., 2010b). Students in Finland take a six hour test that is part of the Matriculation examination. The tests are marked both internally and externally. Massachusetts has a grade 10 exam that all students must complete to gain their school diploma. In addition they must successfully complete the high school mathematics courses (Mathematics I, II, III or Algebra I, II and geometry) in order to apply for college or university.

All of the countries allow calculators in at least part of their assessment. Like Ireland, Finland and Singapore expect them to be used in all the assessments. In Singapore graphing calculators are required for A levels (H1, H2, H3). Examinations comprise a range of item types including multiple choice (Massachusetts), short response (Ireland, New Zealand, Scotland, Singapore) and extended response.

Conclusions

This report is based on the publicly available documentation about curriculum and assessment for upper secondary mathematics education in Ireland, Finland, Massachusetts, New Zealand, Scotland and Singapore. There is considerable commonality around the aims and objectives, curriculum content and progression, and aspirations for problem-solving.

The mathematics expectations in Ireland are comparable to those in the other countries that offer a broad upper secondary curriculum. There are some distinctive features, as discussed in the report. The NCCA may wish to consider the following points in the future development of its Leaving Certificate Mathematics curriculum:

- Should factual recall be the first objective for leaving certificate mathematics?
- Should there be more emphasis on modelling alongside problem-solving?
- Should complex numbers be included at Ordinary level?
- Should speed-time and distance-time graphs be explicitly mentioned?
- Should transformations of functions be explicitly mentioned?
- Should vectors and matrices be introduced into the curriculum?
- Should differential equations be included in the development of calculus?

Obviously any additional content may be at the expense of synthetic geometry, proof and the emphasis on statistics; some distinctive elements that Ireland may consider worth retaining. Some of this additional content, in particular differential equations and vectors and matrices, may be worth considering in the review of the Applied Mathematics qualification.

A wider range of assessment types might allow greater personalisation of the curriculum (as In New Zealand) which may increase participation in advanced mathematics.

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[2] All websites were accessed during March 2013