Algebra in the senior primary classes

Commissioned research paper

Dr Aisling Twohill

Dublin City University

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

June 2020

Introduction

Children's engagement with algebraic thinking has traditionally commenced in senior primary or secondary school, preceded by primary school curricula that prioritised computation and understandings of number (Kieran, et al., 2016). Increasingly during the latter decades of the 20th century, educators and researchers identified that such an approach may contribute to insurmountable challenges for some children when they first encounter formal algebra, typically after six to eight years of school. In this paper I present the research underpinning the Early Algebra movement that arose from a motivation to address such challenges, and the implications of Early Algebra for the Irish Primary School Mathematics Curriculum (IPSMC). While the 1999 IPSMC included algebra as a content strand, key concepts of Early Algebra, such as generalisation, and exploration of structure, are absent (Twohill, 2013).

The importance and relevance of this domain area for children's learning

Before attending to the relevance of the algebra strand to mathematics generally, it is pertinent to unpack what is intended by, and what is understood from, the term 'algebra'. Kieran (2004) identifies two contrasting conceptualisations of algebra, which she labels 'formal algebra' and 'algebraic thinking'. Formal algebra focuses on the application of symbolic expressions to solve problems, and on the manipulation of abstract symbols. In contrast, algebraic thinking includes "analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting" (p. 149). While algebraic thinking may involve the use of abstract symbols as a means of communicating and working with relationships, the focus in algebraic thinking is on the propensity to identify, describe and work with relationships and structure. In this research paper, I draw on an established international body of research that highlights the relevance of algebraic thinking to children attending primary schools, whereby children's innate propensities for algebraic thinking are nurtured into skilful identification and expression of structure, including generalisations (Cai & Knuth, 2011a; Kaput, Carraher, & Blanton, 2008; Kieran et al., 2016).

The Early Algebra movement advocates for increasing the opportunities to develop children's algebraic thinking from early in their education while highlighting that this is not an invitation to move abstract manipulation to earlier in children's education (Cai & Knuth, 2011b; Kaput, 1998; Carpenter, Franke &

Levi, 2003). The role of Early Algebra is to nurture children's growing potential to understand structure. Mason (2008) contends that before commencing school, children already demonstrate the ability to imagine and express, to focus and de-focus, to specialize and generalize, to conjecture and convince, to classify and characterize and that these skills are fundamental to algebraic thinking. As highlighted by Dunphy, Dooley and Shiel (2014) the development of children's mathematical thinking is optimised when mathematical-rich activities build upon children's existing proficiencies, and when teachers focus on the children's reasoning about the mathematics. Resonating with Dunphy et al. is the emphasis of the Early Algebra movement on the priority of conceptual understanding over procedural approaches, along with identification of appropriate representations and contexts to support the developing algebraic thinking of children attending primary school (Kieran et al., 2016).

As children progress through primary school and develop proficiency in identifying and describing structure, opportunities arise for formalisation of their expressions, through the use of abstract symbols. Key content areas of generalised arithmetic and functions (including shape patterns) offer ample opportunities for identification of "what is changing and what is staying the same", and through explorations of structure, children encounter and describe constants, variables, and rates of change (Kieran et al., 2016; Warren & Cooper, 2008). Processes, that have been traditionally associated with abstract equations, may be grounded in sense-making and conceptual understanding. For example, rather than being asked to solve for x from the expression $2+4x=338$, children may be asked to:

> Find an expression for the number of seats in any row in a theatre where the first row contains 6, the next 10, the third 14, the fourth 18, etc. You might find it useful to draw a diagram. Using your expression, work out the number of the row that contains 338 seats.

Mathematics education at primary level plays a central role in supporting children as they develop sophistication in their thinking (Dunphy, et al. 2014). Throughout this paper, attention is paid to the communication of children's thinking, as the efficiency of expressing ideas in abstract symbols is a necessary part of algebraic thinking. In line with a curriculum that strives to develop conceptual understanding by building upon children's thinking, children's work with abstract symbols should a) center around communication of the children's ideas, b) be introduced in response to the children experiencing a need or desire for efficiency, and c) follow after and build upon ample opportunity for children to express their thinking using language that is natural and familiar to them.

The key concepts associated with this domain that children will learn and develop

Blanton et al. (2018) identified four key practices of algebraic thinking as: a) generalising, b) representing, c) justifying, and d) reasoning with generalisations, emphasising that these practices must focus upon structures and relationships. Blanton et al. (2015) presented the following three content areas within which children may apply the key practices of algebraic thinking: a) generalised arithmetic, b) equations, and c) functional thinking. In this section I will unpack each content area in relation to the key practices of algebraic thinking, and identify how children's thinking may be developed within each content area in primary school classrooms.

***Generalised arithmetic and equations*:** Generalising from observed instances to all numbers is a core process of mathematical proficiency and mathematization (Dunphy et al., 2014). This big idea in mathematics has applications at the highest levels of mathematics study and beyond. Notwithstanding the power of generalisation, it is inherently accessible to children of all ages to generalise from their lived experience to understand the structure of the number system and of operations, by considering "indeterminate quantities" (Radford, 2011). For example, junior infants may physically represent zero added to three, and zero added to seven, and be thereafter prompted to consider what the sum of zero and appropriately large (147) or complex (¾) numbers might be. In this case, and for many children of primary school age, generic numbers which are sufficiently large or complex allow children to "distance" their thinking from familiar numbers and think about the generic numbers as place-holders for *all* numbers (Mason & Pimm, 1984; Radford, 2010).

An algebraic approach to generalised arithmetic and equations places the teacher's and child's focus on how relationships unfold (Russell, Schifter & Bastable, 2011). Children are required to explore whether having observed that 2 x 3 = 3 x 2, this pattern holds true for all numbers, whether they can show this by diagram or model, and how they would express this as a rule in general. In a manner analogous to points made later in this paper about the use of abstract symbols, caution is advised in relation to definitions or statements of properties of operations. Children should be afforded opportunities to express their ideas in *their* words, or through the use of representations, with appropriate clarification, modelling and re-voicing from teachers (ibid.)

***Functional thinking*:** Functional thinking embodies an approach that sees functions as descriptions of relationships about how the values of some quantities depend in some way upon the values of other quantities (Chazan 1996). A typical representation of a function that is appropriate for children in primary school is that of a shape pattern (also referred to as a growing pattern, as distinct from a repeating pattern). When working with shape patterns, children are asked to discover or explore a function that relates the number of elements of some component of a pattern figure, to the position of that figure in the pattern. For example, in the following pattern of fences (Figure 1), the number of posts in each fence is a function of the fence number (x), where the specific function is f(x)=3x+1.

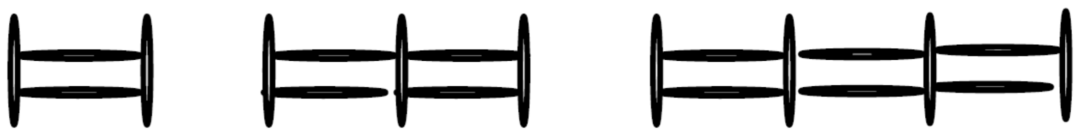

Figure 1. A pattern of fences, wherein the number of posts in each fence is a function of the fence number.

The strategy a child adopts in seeking to generalise from a pattern rests upon his/her observations of figural and numerical aspects of the pattern structure along with observations of relationships and connections within the pattern, and proficiency in multiplicative thinking. Rivera and Becker (2011) highlight the variety of ways in which children and indeed adults “see” patterns. Many are drawn instinctively to comparing consecutive terms (‘recursive thinking’) where some see figure numbers reflected in associated figures of the pattern (‘explicit thinking’) (Lannin, 2005). Some children draw from the structure of figures to support their thinking, while others focus mostly on patterns in the numbers of elements. In supporting children in selecting and applying appropriate strategies for the functions they encounter, teachers have a vital role in facilitating children in accessing multiple approaches. Assumptions should not be made about children’s potential to reason in novel ways when their thinking is mediated by peer interactions, representations including concrete materials and cognitively demanding tasks (Twohill, 2018).

In the fences example above, the opportunity is presented for children to describe a general term for the pattern by marrying the figural and numerical structures of the figures presented (Radford, 2011). As children engage with mathematical-rich patterning activities, and their proficiency in exploring pattern structure and expressing relationships is thereby developed, it will become appropriate to draw their attention to the relationship between elements of patterns that change (variables) and elements that do not change (constants). The validity of expressions of structure should be judged upon their relevance to the pattern, rather than whether the expression is by a long and clumsy sentence, or by using abstract symbols. This does not preclude the use of abstract symbols in expressions, as the role of education is in facilitating children to make progress along a developmental pathway that is appropriately paced for each child, and builds upon where the child is without limiting the child's attainment (Dunphy et al., 2014).

The relationship of the four main processes of the IPSMC with this domain area:

Understanding

Kieran (1996) presents a model for deconstructing 'algebra' into three constituent elements: generational activities, transformational activities and global meta-level activities. Generational activities involve the production of algebraic objects, for example expressions of generality in arithmetic, expressions of generalities in patterns, or expressions containing unknowns that represent problems to be solved. Transformational activities involve manipulation of abstract symbols in order to simplify, expand and/or find solutions. Global meta-level activities involve the use of algebra as a tool within other areas of mathematics, and beyond mathematics, such as "problem solving, modelling, finding structure, justifying, proving and predicting" (p. 272). Traditionally, children have often been presented with entire algebra syllabi that focused on a procedural approach to transformational activities alone, for example following steps to solve for *x*, bringing quantities across the equals sign, etc. However, it is possible for activities to facilitate children in engaging in both generational and transformational activities, or to engage in the global meta-level activity of exploring the underlying mathematical structure of a situation in order to answer conjectural questions (Kieran, 1996). Indeed, it is highly unlikely that children could generate expressions, or engage in problem-solving with the use of expressions without also engaging in, and developing proficiency in transformational activities. The distinction lies in a view of mathematics whereby children learn through rich, meaningful activities rather than mechanical, and sometimes mindless, repetition of procedures.

Connecting

Algebraic thinking holds key affordances for children's work with number, both in terms of their proficiency to perform efficient and mindful computation, along with a conceptual understanding of the place value system. Proficiency in exploring structure and applying observed patterns supports conceptual understanding of units in measurement, and raises awareness of how many properties may be generalisable beyond presented shapes to all shapes within a category.

Communicating

There are many ways in which algebraic thinking may be communicated, other than through mathematical statements containing the abstract symbolism most readily identified with formal algebra (Radford, 2010). Radford (2012) asserts that while an activity may not involve symbols in the expression of ideas, this does not necessarily erode the algebraic nature of the thought processes involved. Kaput, Blanton and Moreno (2008) maintain however, that symbolisation is a core element of algebraic reasoning as it is intrinsic to generalisation. They assert that in the expression of a generalisation, one is speaking about multiple incidences without repetition, and that the use of symbols in so doing is efficient and purposeful. However, Kaput et al. present activities and processes where algebraic thinking is involved but without symbolisation. They term these activities and thought-processes as quasi-algebraic and include expressing generalisations verbally or with concrete objects. Brizuela and Earnest (2008) speak about the language of mathematical symbols, saying that, in general, language is a system through which children learn to communicate their ideas "based on a common set of rules" (p. 274), and likewise, children must learn to use the language system of algebraic symbols to represent their thinking. As with all languages, the acquisition of the language of symbols requires opportunities to express oneself. In planning for a developmental pathway in algebraic thinking it is therefore pertinent to remain cognisant that:

- children will require familiarity with the language system of symbols before they should be expected to reason with them;
- such familiarity should draw on active engagement with mathematical-rich tasks wherein children express their personal observations and speculations in meaningful ways;
- When children are proficient in representing and describing relationships and change verbally, they may build upon this understanding in order to express similar relationships efficiently using abstract symbols, for example:

  Paul is 4 years older than his brother, Ryan. Paul's age is always his brother's age plus four.
  Paul's age=Ryan's age+4; $p = r + 4$.
  The number of tiles needed for a square floor is the number of tiles along the side multiplied by itself; (Number of tiles along one side)x(Number of tiles along that side); $n \times n$; n^2.

Mathematical-rich tasks, such as shape patterning, afford children opportunities to engage in transformational algebraic activity that is grounded in sense-making and supported by the child's observations of the structure of the pattern. For example, Rivera and Becker (2011) emphasise how people tend to see patterns in a variety of ways, and Warren and Cooper (2008) highlight how this variety affords opportunities for children to compare equivalent expressions.

Reasoning

Dooley, Dunphy and Shiel (2014) draw on the work of Reid (2002) to highlight three core elements of reasoning as identifying patterns, proving or disproving their observations, and explaining why. These three core elements are intrinsic to algebraic thinking, both in generalised arithmetic and functional thinking, as children build upon their observations of structure to present conjectures for generalities, which they must then be encouraged to check and prove systematically. Lannin (2005) emphasises that "generalization cannot be separated from justification" (p. 235), where justification includes proving or disproving and explaining "why". Lannin outlines a framework for children's development of rigorous justification skills. Children commence at a level where they use no justification, progress to a level where they appeal to a higher authority, and from this position advance to a level where the child is capable of demonstrating empirically why their generalisation should be held to be true.

Applying and problem-solving

As highlighted above, while algebraic thinking is a rich mathematical domain wherein children may engage in reasoning about abstractions in accessible ways, there remains a risk of over-scaffolding and thus removing the challenge through which children learn in engaging and rewarding activities. Sullivan, Knott, & Yang (2015) emphasise the potential of tasks to either (a) facilitate discovery within specific mathematical content, or (b) identify to learners the target content at the beginning of the lesson, thus removing the potential for discovery learning. To best support children in thinking algebraically, teaching methods will support children in exploring and describing patterns and structure, without pre-emptive moves on the part of the teacher. For example, teachers may have encountered approaches to the solution of functions, wherein one is directed to look to the coefficient of the variable as the rate of change. In a manner similar to children learning about multiplication of fractions, observations of relationships between coefficients and rates of change will be useful and applied correctly when the child understands the relationship, and has discovered it for him/herself. The value inherent in children constructing understanding in ways such as this is largely accepted in mathematics education research, but teaching approaches underpinned by transmission persist in many classrooms in Ireland (Dooley, 2011; Nic Mhuirí, 2013).

Key messages

Research conducted in 2015 with children attending Irish primary schools demonstrated that many children are capable of thinking in novel and creative ways about the structure of functions, and that generalisation as a high order mathematical concept is accessible to children in primary school in Ireland (Twohill, 2018). It is pertinent to emphasise that the algebraic thinking demonstrated in the 2015 study was facilitated by a problem-solving approach mediated by peer group interactions and concrete materials. In a pilot study of individual assessments of algebraic thinking using a paper-and-pencil format, the children's engagement was characterised by uncertainty and an absence of suggestions (ibid.). English (2011) warns that teachers and policy makers should not underestimate children's ability to take on and work with new ways of thinking. English states that children "have access to a range of powerful ideas and processes and can use these effectively to solve many of the mathematical problems they meet in daily life" (p. 491).

Algebra has traditionally been associated with abstraction, and an insider language of abstract symbols that for some children served to exclude them from advancing in mathematics (Mason, 2008; Lakoff & Nunes, 2003). The presentation of algebraic thinking within this paper advocates for an understanding of algebra that is both more powerful and more accessible, as all children are afforded access to expressing mathematical ideas abstractly, by building upon their natural language to express their personal observations (Kieran et al., 2016).

Glossary

Generalising: To make assertions, claims or justifications as to how understanding is applicable or transferrable to other circumstances.

Indeterminate quantity: A quantity which has no fixed value, but which may be varied in accordance with any proposed condition.

Re-voicing: The teacher repeats some or all of what the child has said and then asks the child to clarify whether or not this may be correct.

Shape Pattern: Also referred to as a growing pattern, occurs when a group of shapes are repeated over and over again.

References

Blanton, M., Brizuela, B. M., Murphy Gardiner, A., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*(5), pp. 511- 558.

Blanton, M., Brizuela, B. M., Stephens, A., Knuth, E., Isler, I., Gardiner, A. M., & Stylianou, D. (2018). Implementing a framework for early algebra. In Kieran, C. (Ed.), *Teaching and Learning Algebraic Thinking with 5-to 12-Year-Olds.* (pp. 27-49). Springer, Cham.

Brizuela, B. M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the "best deal" problem. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades.* (pp. 273-302). New York: Lawrence Erlbaum Ass.

Cai, J., & Knuth, E. (2011a). *Early algebraization: A global dialogue from multiple perspectives.* Heidelberg: Springer.

Cai, J., & Knuth, E. (2011b). Introduction: a global dialogue about early algebraization from multiple perspectives. In J. Cai, & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives.* (pp. vii-xi). Heidelberg: Springer.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* USA: Heinemann.

Chazan, D. (1996). Algebra for all students. *Journal of Mathematical Behavior, 15*(4), 455–477.

Dooley, T., (2011). Telling matters in mathematics teaching and learning. In T. Dooley, D. Corcoran & M. Ryan (Eds.), *Proceedings of the Fourth Conference on Research in Mathematics Education (MEI 4)*. (pp. 19-38). Dublin: St Patrick's College.

Dooley, T., Dunphy, E., & Shiel, G. (2014). *Mathematics in early childhood and primary education (3-8 years): Volume 2: Teaching and learning.* Dublin: NCCA.

Dunphy, E., Dooley, T., & Shiel, G. (2014). *Mathematics in early childhood and primary education (3-8 years). Volume 1: Definitions, theories, development and progression.* Dublin: NCCA.

English, L. (2011). Complex learning through cognitively demanding tasks. *The Mathematics Enthusiast, 8*(3), 483-506.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In S. Fennell (Ed.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades.* (pp. 19-56). New York: Lawrence Erlbaum Ass.

Kaput, J., Carraher, D. W., & Blanton, M. L. (2008). *Algebra in the early grades*. New York: Lawrence Erlbaum Ass.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. M. Alvares, B. Hodgson, C. Laborde and A. Perez (Eds.), *Proceedings of the 8th International Congress on Mathematical Education: selected lectures.* (pp. 271-290). Sociedad Andaluza de Educación Matemática 'Thales': Seville.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator, 8*(1), 139-151.

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). Early Algebra: Research into its Nature, its Learning, its Teaching. In G. Kaiser, (Ed.) *ICME-13 Topical Surveys*. Springer Open. doi:10.1007/978-3-319-32258-2.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* AMC, 10(12), 720-733.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7*(3), 231-258.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 57-94). New York: Lawrence Erlbaum Ass.

Mason, J., & Pimm, D. (1984). Generic Examples: Seeing the General in the Particular.*Educational Studies in Mathematics, 15*(3), 277-289.

Nic Mhuirí, S. (2013). A mathematical discourse community in an Irish primary classroom. In T. Dooley, S. NicMhuirí, M. OReilly & R. Ward (Eds.), *Proceedings of the Fifth Conference on Research in Mathematics Education (MEI 5).* (pp. 236-247). Dublin: St Patrick's College.

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12* (1), 1-19.

Radford, L. (2011). Students' non-symbolic algebraic thinking. In J. Cai, & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives.* (pp. 303-322). Heidelberg: Springer.

Radford, L. (2012). Early algebraic thinking: Epistemological, semiotic, and developmental issues. *Regular lecture presented at the 12th International Congress on Mathematical Education,* Seoul, Korea. Retrieved December 13, 2012, from http://www.luisradford.ca/pub/5_2012ICME12RL312.pdf.

Reid, D. A. (2002). Conjectures and refutations in Grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5–29.

Rivera, F., & Becker, J. R. (2011). Formation of pattern generalization involving linear figural patterns among middle school students: Results of a three-year study. In J. Cai, & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives.* (pp. 323-366). Heidelberg: Springer.

Russell, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai, & E. Knuth (Eds.), *Early algebraization* (pp. 43-69). Springer, Berlin, Heidelberg.

Schifter, D., Bastable, V., Russell, S. J., Seyferth, L., & Riddle, M. (2008). Algebra in the grades K-5 classroom: Learning opportunities for students and teachers. In C. E. Greenes, & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics.* (pp. 263-278). Reston, VA: NCTM.

Sullivan, P., Knott, L., & Yang, Y. (2015). The relationships between task design, anticipated pedagogies, and student learning. In A. Watson & M. Obtain (Eds.), *Task design in mathematics education, an ICMI study 22* (pp. 83-114). Switzerland: Springer, Cham.

Twohill, A. (2013). Exploring the role of patterns: Early algebra in the Irish primary mathematics curriculum. In: T. Dooley, S. Nic Mhuirí, M. O'Reilly & R. Ward, (Eds.). *Proceedings of the Fifth Conference on Research in Mathematics Education.* (pp.332-343). Dublin: St Patrick's College.

Twohill, A. (2018). Observations of structure within shape patterns. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice*. (pp. 213-236). Springer: Berlin.

Warren, E., & Cooper, T. J. (2008). Patterns that support early algebraic thinking in the elementary school. In C. E. Greenes, & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics.* (pp. 113-126). Reston, VA: NCTM.

NCCA